Ideas on Schwarz Smoothers in Efficient Multigrid Solvers

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deal.II

IWR
Interdisciplinary Center for Scientific Computing

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Problem

A typical finite element problem reads in weak form

$$a(u_h, v_h) = (f, v_h) \qquad \forall v_h \in V_h$$

Discretization $\Rightarrow Ax = b$.

Solving large problems is restricted by

- Memory limitations
  - $\Rightarrow$ Matrix-Free methods
- Time limitations (Scalability)
  - $\Rightarrow$ Parallelization
  - $\Rightarrow$ Geometric Multigrid

Matrix-Free

Iterative solvers only require a matrix-vector product.
$\Rightarrow$ Idea:

- Storing the matrix not required
- "Assemble" the matrix-vector product directly, when solving.[1]

Matrix-free algorithm:

- v=0
- loop over cells
  1. Extract local vector values on cell: $u_K = P_K u$
  2. Apply operation locally on cell: $v_K = A_K u_K$ (without forming A_K)
  3. Sum results into the global solution vector: $v = v + P_K^T v_K$

[1] Martin Kronbichler and Katharina Kormann. "A generic interface for parallel cell-based finite element operator application". In: *Computers & Fluids* 63 (2012), pp. 135 –147. ISSN: 0045-7930

Matrix-Free

Matrix-Based

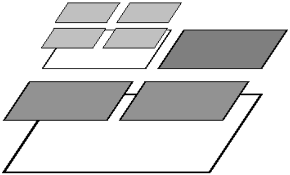
1. Build matrix $A^K_{i,j} = \int_K a(\phi_j, \phi_i)\, dx$
2. Solve using iterative method (CG) $\rightarrow$ compute $v_i = (Au)_i$.

Matrix-Free

Compute matrix-vector product directly

$$v_i = (Au)_i = \sum_K \int_K a(u, \phi_i)\, dx$$

Parallel Adaptive Multigrid



- Only method known that is $\mathcal{O}(N)$
- Geometric multigrid on adaptively refined meshes with local smoothing[2]
- `deal.II` implements parallel geometric multigrid for both continuous and discontinuous elements.

[1] Bärbel Janssen and Guido Kanschat. "Adaptive Multilevel Methods with Local Smoothing for H^1- and H^{curl}-Conforming High Order Finite Element Methods". In: *SIAM Journal on Scientific Computing* 33.4 (2011), pp. 2095–2114

Parallel Adaptive Multigrid - V-Cycle

1. Pre-smoothing:

$$u^{(k+1)} = u^{(k)} - R_l^{-1}(A_l u^{(k)} - f_l), \qquad 0 \leq k < m_{pre}$$

2. Coarse grid correction:

$$\begin{aligned} f_{l-1} &= \Pi_{l-1}^T (f_l - A_l u^{(m_{pre})}) \\ v^{(k+1)} &= MG_{l-1}(v^{(k)}, f_{l-1}), \qquad 0 \leq k < m_{coarse} \\ w^{(0)} &= u^{(m_{pre})} + v^{(m_{coarse})} \end{aligned}$$

3. Post-smoothing:

$$w^{(k+1)} = w^{(k)} - R_l^{-1}(A_l w^{(k)} - f_l), \qquad 0 \leq k < m_{post}$$

4. Assign: $MG(u^{(0)}, f_l) = w^{(m_{post})}$

Coarse grid solver $MG_0(u(0), f) = A_0^{-1} f_0$

Parallel Adaptive Multigrid - V-Cycle

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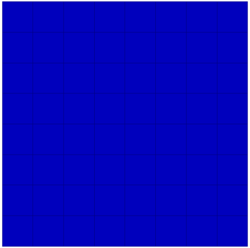
Additive Schwarz Smoother

Hermann Amandus Schwarz

- Take the local structure of the problem into account
- Use local problems for preconditioning

$$R_l = \sum_{K \in \mathcal{T}_l} P_K A_K^{-1}$$



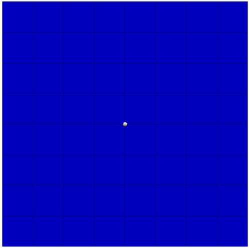
Additive Schwarz Smoother

Hermann Amandus Schwarz

- Take the local structure of the problem into account
- Use local problems for preconditioning

$$R_l = \sum_{K \in \mathcal{T}_l} P_K A_K^{-1}$$

- Point patches ⇒ Jacobi



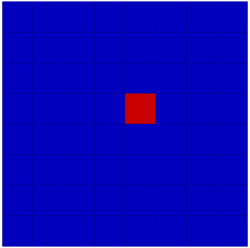
Additive Schwarz Smoother

Hermann Amandus Schwarz

- Take the local structure of the problem into account
- Use local problems for preconditioning

$$R_l = \sum_{K \in \mathcal{T}_l} P_K A_K^{-1}$$

- Cell patches ⇒ BlockJacobi



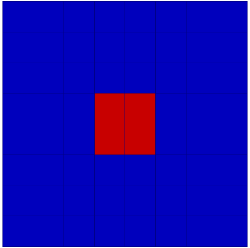
Additive Schwarz Smoother

Hermann Amandus Schwarz

- Take the local structure of the problem into account
- Use local problems for preconditioning

$$R_l = \sum_{K \in \mathcal{T}_l} P_K A_K^{-1}$$

- Vertex patches $\Rightarrow$ BlockJacobi



Abstract Schwarz Theory

PDE problem

Let V be Hilbert space, $f \in V'$. Find $u \in V$ such that

$$a(u, v) = f(v) \quad \forall v \in V,$$

where $a(.,.)$ is a symmetric, positive definite (s.p.d.) bilinear form.

Schwarz Operators

Assume V admits decomposition: $V = R_0^T V_0 + \sum_{j=1}^{J} R_j^T V_j$. Then, Schwarz operators $P_j \equiv R_j^T \tilde{P}_j$ are defined by projection-like operators $\tilde{P}_j : V \to V_j$:

$$\tilde{a}_j(\tilde{P}_j u, v_j) = a(u, R_j^T v_j) \quad \forall v_j \in V_j,$$

where $\tilde{a}_j(.,.)$ are local bilinear forms (s.p.d.).

Convergence Theory

I. Stable Decomposition

It exists C_0 such that every $u \in V$ has a decomposition $u = \sum_{j=0}^{J} R_j^T u_j$ that satisfies $\sum_{j=0}^{J} \tilde{a}_j(u_j, u_j) \leq C_0^2 a(u, u)$.

II. Strengthened Cauchy-Schwarz Inequality

It exist $0 \leq \epsilon_{ij} \leq 1, 1 \leq i, j \leq J$, such that

$$|a(R_i^T u_i, R_j^T u_j)| \leq \epsilon_{ij} a(R_i^T u_i, R_i^T u_i)^{1/2} a(R_j^T u_j, R_j^T u_j)^{1/2},$$

for $u_i \in V_i$ and $u_j \in V_j$. Define $\mathcal{E} = (\epsilon_{ij})$.

III. Local Stability

There exists $\omega > 0$, such that

$$a(R_j^T u_j, R_j^T u_j) \leq \omega \tilde{a}_j(u_j, u_j), \; u_j \in range(\tilde{P}_j) \subset V_j, \; 0 \leq j \leq J.$$

Convergence Theory

Theorem

Let Assumptions I. - III. be satisfied. Then

$$\kappa(P_{ad}) \leq C_0^2 \omega(\rho(\mathcal{E}) + 1).$$

- Schwarz operator: $P_j = R_j^T \tilde{A}_j^{-1} R_j A, \ j = 0, \dots, J$
- additive preconditioner:

$$P_{\text{ad}} = A_{\text{ad}}^{-1} A = \sum_{j=0}^{J} \left(R_j^T \tilde{A}_j^{-1} R_j A \right)$$

- multiplicative preconditioner:

$$P_{\text{mu}} = A_{\text{mu}}^{-1} A = I - \prod_{j=0}^{J} \left(I - R_j^T \tilde{A}_j^{-1} R_j A \right)$$

- mixtures possible

The Stokes problem

Stokes

Consider the Stokes problem

$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (p, \nabla \cdot \boldsymbol{v}) + (\nabla \cdot \boldsymbol{u}, q) + \tau_{gd}(\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})$$
$$\nabla \cdot \boldsymbol{u} = 0$$

with the grad-div stabilization term $\tau_{gd}(\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v})$.

This leads to the following structure of the system matrix:

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ 0 \end{pmatrix}$$

Hence, we have to solve a symmetric , but indefinite problem.

Raviart-Thomas Elements

Original result from Kanschat and Mao[3] using Raviart-Thomas elements.

Key assumption

$$\nabla \cdot V_h = Q_h \qquad V_{h,0}^{div} \subset \ldots \subset V_{h,L}^{div}$$

where

$$V_{h,l}^{div} := \{v_h \in V_l : (\nabla \cdot u_h, q_h) = 0 \quad \forall q_h \in Q_h\}$$

Can this assumption be weakened and the result be applied to other inf-sup stable elements?

[3] Guido Kanschat and Youli Mao. "Multigrid methods for Hdiv-conforming discontinuous Galerkin methods for the Stokes equations". In: *Journal of Numerical Mathematics* 23.1 (2015), pp. 51–66

Raviart-Thomas Elements

Theorem[4]

The multilevel iteration $I - \mathcal{B}_L \mathcal{A}_L$ for the Stokes problem

- with the variable V-cycle operator $\mathcal{B}_L$
- employing the smoother $\mathcal{R}_l$ with suitably small scaling factor η

is a contraction with contraction number independent of the level L.

[4] Guido Kanschat and Youli Mao. "Multigrid methods for Hdiv-conforming discontinuous Galerkin methods for the Stokes equations". In: *Journal of Numerical Mathematics* 23.1 (2015), pp. 51–66

Stokes, Perturbed Problem

Idea: Eliminate the pressure by considering a perturbed formulation

$$\nu(\nabla \boldsymbol{u}_l, \nabla \boldsymbol{v}_h) + \tau_{gd}(\nabla \cdot \boldsymbol{u}_h - \epsilon p_h, \nabla \cdot \boldsymbol{v}_h - \epsilon q_h)$$
$$-(p_l, \nabla \cdot \boldsymbol{v}_h) + (\nabla \cdot \boldsymbol{u}_l, q_h) - \epsilon(p_l, q_h) = (\boldsymbol{f}, \boldsymbol{v}_h)$$

Defining the operator $\mathcal{A}_l : \boldsymbol{V}_l \times Q_l \to (\boldsymbol{V}_l \times Q_l)^*$ by

$$\begin{aligned}\mathcal{A}_l((\boldsymbol{u}_l, p_l), (\boldsymbol{v}_h, q_h)) :=& \nu(\nabla \boldsymbol{u}_l, \nabla \boldsymbol{v}_h) \\ &+ \tau_{gd}(\nabla \cdot \boldsymbol{u}_l - \epsilon p_l, \nabla \cdot \boldsymbol{v}_h - \epsilon q_h) \\ &+ (p_l, \nabla \cdot \boldsymbol{v}_h) + (\nabla \cdot \boldsymbol{u}_l, q_h) - \epsilon(p_l, q_h).\end{aligned}$$

this problem can be written as $\mathcal{A}_l((\boldsymbol{u}_l, p_l), (\boldsymbol{v}_h, q_h)) = (\boldsymbol{f}, \boldsymbol{v}_h)$ for all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_l \times Q_l$.

Stokes, Perturbed Problem

For $\epsilon > 0$, the Stokes problem can be rewritten as

$$\begin{aligned}\mathcal{A}_l(\boldsymbol{u}_l, \boldsymbol{v}_h) := \nu(\nabla \boldsymbol{u}_l, \nabla \boldsymbol{v}_h) \\ + \tau_{gd}(\pi_{Q_h}^{\perp}(\nabla \cdot \boldsymbol{u}_l), \pi_{Q_h}^{\perp}(\nabla \cdot \boldsymbol{v}_h)) \\ + \frac{1}{\epsilon}(\pi_{Q_h}(\nabla \cdot \boldsymbol{u}_l), \pi_{Q_h}(\nabla \cdot \boldsymbol{v}_h)).\end{aligned}$$

$$\mathcal{A}_l(\boldsymbol{u}_l, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_l$$

Lemma

Let $(\boldsymbol{u}_l, p_l)$ be the solution to the perturbed problem in two variables and $\widetilde{\boldsymbol{u}}_l$ the solution to the perturbed problem in one variable. Then it holds

$$\widetilde{\boldsymbol{u}}_l = \boldsymbol{u}_l \qquad \epsilon p_l = \pi_{Q_h}(\nabla \cdot \boldsymbol{u}_l) = \pi_{Q_h}(\nabla \cdot \widetilde{\boldsymbol{u}}_l)$$

Convergence of the Perturbation

Let $(\boldsymbol{u}, p)$ be the solution to the continuous Stokes problem and $(\boldsymbol{u}_h, p_h)$ the solution to the discretized (perturbed) problem.

Lemma

It holds

$$\nu\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_0^2 + \tau_{gd}\|\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h)\|_0^2 + \|p-p_h\|_0^2$$
$$\lesssim \epsilon + h^{2k_p+2} + h^{2k_u}.$$

Convergence Result

Convergence

If $\mathcal{R}_l$ satisfies for all $\boldsymbol{w} \in \boldsymbol{V}_l$

$$\mathcal{A}_l((\mathcal{I}_l - \mathcal{R}_l\mathcal{A}_l)\boldsymbol{w}, \boldsymbol{w}) \geq 0 \tag{1}$$

$$(\mathcal{R}_l^{-1}[\mathcal{I}_l - \mathcal{P}_{l-1}]\boldsymbol{w}, [\mathcal{I}_l - \mathcal{P}_{l-1})\boldsymbol{w}) \leq \beta_l\mathcal{A}_l([\mathcal{I}_l - \mathcal{P}_{l-1}]\boldsymbol{w}, [\mathcal{I}_l - \mathcal{P}_{l-1}]\boldsymbol{w}), \tag{2}$$

then it holds

$$0 \leq \mathcal{A}_l([\mathcal{I}_l - \mathcal{B}_l\mathcal{A}_l)\boldsymbol{w}, \boldsymbol{w}) \leq \delta\mathcal{A}_l(\boldsymbol{w}, \boldsymbol{w}), \quad \forall \boldsymbol{w} \in \boldsymbol{V}_l$$

where $\delta < 1$ is independent of the number of levels.

Lemma

Let $\eta \leq 2^{-dim}$*, then*

$$\mathcal{A}_l((\mathcal{I}_l - \mathcal{R}_l\mathcal{A}_l)\boldsymbol{w}, \boldsymbol{w}) \geq 0, \quad \forall \boldsymbol{w} \in \boldsymbol{V}_l.$$

Stable Decomposition

Lemma

It holds

$$\eta \sum_v (\mathcal{R}_l^{-1} \boldsymbol{u}, \boldsymbol{u}) = \inf_{\substack{\boldsymbol{u}_v \in \boldsymbol{V}_{l,v} \\ \sum_v \mathcal{I}_{l,v} \boldsymbol{u}_v = \boldsymbol{u}}} \sum_v \mathcal{A}_l(\mathcal{I}_{l,v} \boldsymbol{u}_v, \mathcal{I}_{l,v} \boldsymbol{u}_v)$$

Essentially, we only need to find a decomposition $(\boldsymbol{u}_v)_v$ of $[\mathcal{I}_l - \mathcal{P}_{l-1}]\boldsymbol{w}$, i.e.

$$\boldsymbol{u} := [\mathcal{I}_l - \mathcal{P}_{l-1}]\boldsymbol{w} = \sum_v \mathcal{I}_{l,v} \boldsymbol{u}_v.$$

such that

$$\sum_v (\mathcal{A}_l \mathcal{I}_{l,v} \boldsymbol{u}_v, \mathcal{I}_{l,v} \boldsymbol{u}_v) \leq \beta_l \mathcal{A}_l([\mathcal{I}_l - \mathcal{P}_{l-1}]\boldsymbol{w}, [\mathcal{I}_l - \mathcal{P}_{l-1}]\boldsymbol{w}).$$

Helmholtz-like decomposition

Denote the bilinear form a_l corresponding to the weak Laplace operator by

$$a_l(\boldsymbol{u}, \boldsymbol{v}) := \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v})$$

For $\boldsymbol{u}_l \in \boldsymbol{V}_l$ define $\boldsymbol{u}_l^0 \in \boldsymbol{V}_{h,l}^{div}$ as projection of $\boldsymbol{u}_l$ onto $\boldsymbol{V}_{h,l}^{div}$ with respect to a_l, i.e.

$$a_l(\boldsymbol{u}_l^0, \boldsymbol{v}_l) = a_l(\boldsymbol{u}_l, \boldsymbol{v}_l) \quad \forall \boldsymbol{v}_l \in \boldsymbol{V}_{h,l}^{div}.$$

Then define $\boldsymbol{u}_l^\perp$ by $\boldsymbol{u}_l^\perp := \boldsymbol{u}_l - \boldsymbol{u}_l^0$.

Lemma

$$\frac{\alpha}{d^2}\|\nabla \cdot \boldsymbol{u}_l^\perp\|_0^2 \leq a_l(\boldsymbol{u}_l^\perp, \boldsymbol{u}_l^\perp) \leq \frac{\nu}{\gamma^2}\|\pi_{Q_h}(\nabla \cdot \boldsymbol{u}_l^\perp)\|_0^2$$

Stable Decomposition

Theorem

For any $\mathbf{v}_l \in \mathbf{V}_l$ there exists a decomposition $\mathbf{v}_{l,j}$ such that

$$\sum_{j=0}^{J} \mathcal{A}_l(\mathbf{v}_{l,j}, \mathbf{v}_{l,j}) \lesssim \mathcal{A}_l(\mathbf{v}_l, \mathbf{v}_l)$$

provided $\tau_{gd} \lesssim \min\{\nu, \epsilon^{-1}\}$.

Assumption

$$\sum_{v} a_l(\mathbf{u}_v^\perp, \mathbf{u}_v^\perp) \leq C a_l(\mathbf{u}_l^\perp, \mathbf{u}_l^\perp)$$

This holds for discontinuous, divergence-free elements.
What about $\mathbb{Q}_k/\mathbb{Q}_{k-1}$, $\mathbb{Q}_k/\mathbb{P}^-_{k-1}$ or other inf-sup stable element pairs?

Numerical Results - Test Problem

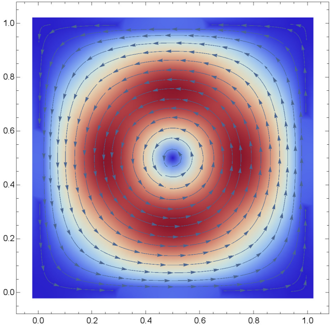
We consider the test problem

$$-\nu\Delta \boldsymbol{u} + \nabla p = -\nu\Delta \boldsymbol{u}_{ref} + \nabla p_{ref}$$
$$\nabla \cdot \boldsymbol{u} = 0$$

with the reference solution

$$\boldsymbol{u}(x,y) = \begin{pmatrix} \sin^2(\pi x)\sin(2\pi y) \\ -\sin^2(\pi y)\sin(2\pi x) \end{pmatrix}$$
$$p(x,y) = \sin(\pi x)\cos(\pi y).$$

in 2D.



Observe for $\nu = 10^{-6}$

- errors
- iteration counts (error reduction by 10^{-6}).

Numerical Results - Error Estimates

Theorem

For a given $\mathbf{f} \in H^{-1}(\Omega)^d$, *let* $(\boldsymbol{u}, p)$ *be the continuous problem, and let* $(\boldsymbol{u}_h, p_h)$ *be the solution to the discrete solution. Then, the error in the energy norm for the velocity is bounded by*

$$\nu\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_0^2 \leq \inf_{\boldsymbol{w}_h \in [H_0^1(\Omega)]^d} \left(4\nu\|\nabla(\boldsymbol{u} - \boldsymbol{w}_h)\|_0^2 + 2\tau_{gd}\|\nabla \cdot \boldsymbol{w}_h\|_0^2\right) + \frac{2}{\tau_{gd}} \inf_{q_h \in Q_h} \|p - q_h\|_0^2.$$

[4] Eleanor W Jenkins et al. "On the parameter choice in grad-div stabilization for the Stokes equations". In: *Advances in Computational Mathematics* 40.2 (2014), pp. 491–516

Numerical Results - Optimal Stabilization Parameter

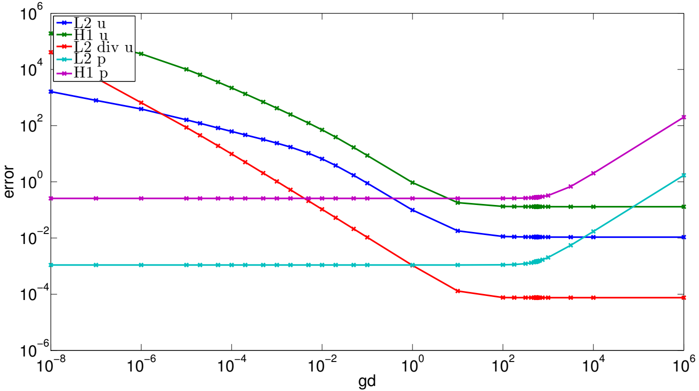


Figure: $\mathbb{Q}_2/\mathbb{Q}_1$ elements, optimal $\tau_{gd} = 544.917$

Numerical Results - Optimal Stabilization Parameter

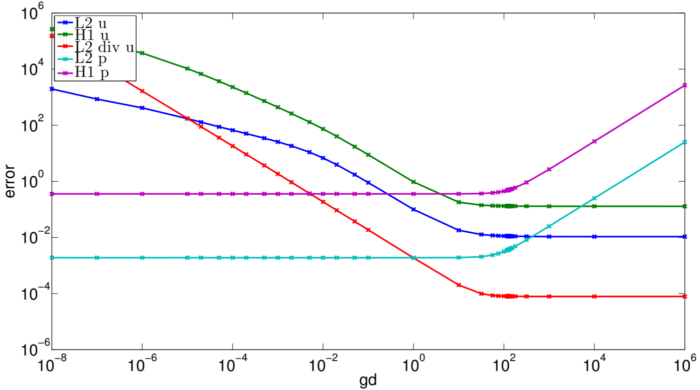


Figure: $\mathbb{Q}_2/\mathbb{P}_1^-$ elements, optimal $\tau_{gd} = 128.93$

Numerical Results

Element	Refinement 1	2	3	4	5	6
$\mathbb{Q}_2/\mathbb{P}_1^-$	$1 \cdot 10^0$	$3 \cdot 10^1$	$2 \cdot 10^1$	$5 \cdot 10^1$	$1 \cdot 10^2$	$9 \cdot 10^1$
$\mathbb{Q}_2/\mathbb{Q}_1$	$9 \cdot 10^0$	$9 \cdot 10^1$	$9 \cdot 10^1$	$2 \cdot 10^2$	$5 \cdot 10^2$	$4 \cdot 10^2$
$\mathbb{Q}_3/\mathbb{P}_2^-$	$1 \cdot 10^5$	$5 \cdot 10^3$	$4 \cdot 10^0$	$1 \cdot 10^1$	$5 \cdot 10^2$	$9 \cdot 10^{-1}$
$\mathbb{Q}_3/\mathbb{Q}_2$	$1 \cdot 10^5$	$8 \cdot 10^2$	$6 \cdot 10^1$	$4 \cdot 10^1$	$4 \cdot 10^2$	$5 \cdot 10^{-1}$

Table: Optimal stabilization parameter

#Levels	0	1	2	3	4
RT1, 2D	3	9	10	11	13
RT2, 2D	3	9	10	11	11

Table: Iteration counts for Raviart-Thomas elements

Numerical Results - Iteration Counts - $\mathbb{Q}_2/\mathbb{P}_1^-$ - $\eta = \frac{1}{4}$ - 2D

	τ_{gd}				
#Levels	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}
0	2	2	2	2	2
1	24	23	27	22	17
2	70	65	57	43	25
3	236	159	93	48	27
4	459	247	105	49	28

	τ_{gd}				
#Levels	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}
0	2	2	2	2	2
1	14	12	13	13	13
2	19	18	18	18	18
3	19	19	19	19	19
4	19	18	19	20	20

Numerical Results - Iteration Counts - $\mathbb{Q}_2/\mathbb{Q}_1$ - $\eta = \frac{1}{8}$ - 2D

	τ_{gd}				
#Levels	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}
0	2	2	2	2	2
1	35	31	36	28	22
2	137	98	85	59	31
3	454	294	159	71	37
4	-	610	190	76	38

	τ_{gd}				
#Levels	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}
0	2	2	2	2	2
1	17	17	17	17	17
2	23	24	24	25	25
3	28	31	34	35	35
4	28	33	38	39	39

Numerical Results - Iteration Counts - $\mathbb{Q}_3/\mathbb{P}_2^-$ - $\eta = \frac{1}{4}$ - 2D

	τ_{gd}				
#Levels	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}
0	2	2	2	2	2
1	12	12	13	14	15
2	19	19	19	20	17
3	30	30	29	24	16
4	39	38	35	24	16

	τ_{gd}				
#Levels	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}
0	2	2	3	3	2
1	15	16	16	16	16
2	15	17	18	18	18
3	15	17	18	18	18
4	14	16	18	18	18

Numerical Results - Iteration Counts - $\mathbb{Q}_3/\mathbb{Q}_2$ - $\eta = \frac{1}{8}$ - 2D

	τ_{gd}				
#Levels	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}
0	2	2	2	2	2
1	19	19	20	20	21
2	31	31	31	27	25
3	36	35	33	32	25
4	50	50	50	36	25

	τ_{gd}				
#Levels	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}
0	2	2	2	2	2
1	21	22	22	22	22
2	25	28	31	31	31
3	26	30	33	34	34
4	27	32	37	37	38

Test case - Jumping Coefficients

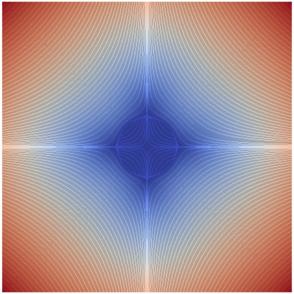


Figure: Reference velocity

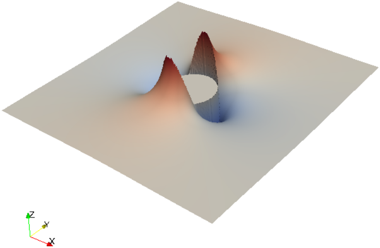


Figure: Reference pressure

$$\nu(r) = \begin{cases} \eta & \text{for } r^2 < r_i^2 \\ 1 & \text{for } r^2 \geq r_i^2 \end{cases}$$

Test case - Jumping Coefficients

$$A = \frac{\eta - 1}{\eta + 1} \qquad z = x + \mathrm{i}y$$

$$\phi = \begin{cases} 0 & \text{for } r^2 < r_i^2 \\ -2Ar_i^2/z & \text{for } r^2 \geq r_i^2 \end{cases} \qquad \psi = \begin{cases} -4z\frac{\eta}{1+\eta} & \text{for } r^2 < r_i^2 \\ \frac{-2(z+Ar_i^4}{z^3} & \text{for } r^2 \geq r_i^2 \end{cases}$$

$$\nu(r) = \begin{cases} \eta & \text{for } r^2 < r_i^2 \\ 1 & \text{for } r^2 \geq r_i^2 \end{cases}$$

$$\boldsymbol{u} = \begin{pmatrix} \Re(\phi - z\overline{\phi'(z)} - \overline{\psi(z)}) \\ \Im(\phi - z\overline{\phi'(z)} - \overline{\psi(z)}) \end{pmatrix} /2\nu(r)$$

$$p = -2\Re(\phi'(z));$$

Test case - Jumping Coefficients

	η					
#level	10		100		1000	
	$\mathbb{Q}_2/\mathbb{P}_1^-$	$\mathbb{Q}_3/\mathbb{P}_2^-$	$\mathbb{Q}_2/\mathbb{P}_1^-$	$\mathbb{Q}_3/\mathbb{P}_2^-$	$\mathbb{Q}_2/\mathbb{P}_1^-$	$\mathbb{Q}_3/\mathbb{P}_2^-$
1	1	1	1	1	1	1
2	1	3	1	3	1	3
3	10	12	10	13	12	16
4	14	14	15	14	21	20
5	15	14	18	15	29	20
6	16	15	19	17	33	25
7	16	15	22	18	40	33
8	16	14	22	19	45	33

Table: #iterations

Dictionary of Inverses

- Theory: We don't have to solve exact local problems!
- Idea: Store only a few local inverses that lead to similar behavior as an exact Schwarz smoother

Check Inverse Similarity

Store only local inverses that differ significantly,

$$\frac{\|A_i^{-1}A_j - I\|_{Frob}}{N_j} < \text{TOL} \qquad \Rightarrow \qquad A_j^{-1} \approx A_i^{-1},$$

where N_j is the amount of DoFs on j-th subdomain.

- use linear elements to check similarity
- generic

Test Settings

- SIPG discretization for diffusion term
- uniform refinement: $c(\Omega_0)4^\ell$ cells on level ℓ
  - unit square: $c = 1$
  - unit ball: $c = 5$
- standard V-cycle with 1 pre- & post-smoothing step
- non-overlapping, additive Schwarz smoother
- residual reduction of 10^{-6}

Laplace - Iterations 2D

$$-\Delta u = 1 \text{ in } (0,1)^2, \quad g_D \equiv 0.$$

	$k = 10$			$k = 7$		
#levels	1-Inv	BJ	Dict(0.01)	1-Inv	BJ	Dict(0.01)
5	19	19	19	18	18	18
6	19	19	19	18	18	18
7	20	20	20	19	19	19
8	21	21	21	19	19	19
9	22	21	21	20	20	20
10	22	22	22	20	20	20

CG–iterations: $\mathbb{Q}_k^-$ elements; Dictionary (4x corner, 4x side, 1x interior); 1-Inverse.

Laplace - Iterations 2D - Unit Ball

$$-\Delta u = 1 \text{ in } B(0,1), \quad g_D \equiv 0.$$

	$k=4$			$k=7$				
#levels	BJ	Dict(0.05)	Dict(0.01)	BJ	Dict(0.05)	(%)	Dict(0.01)	(%)
5	22	25	25	32	36		33	
6	24	28	26	34	43		35	
7	25	32	27	36	49		37	
8	26	34	28	37	54	(0.2)	38	(4)
9	28	37	29	39	61	(0.05)	40	(1)

(*relative inverses on finest level*)

CG–iterations: $\mathbb{Q}_k^-$ elements; 1-Inverse does not converge.

Diffusion - Iterations 2D

$$-\nabla \cdot (D\nabla u) = 1 \text{ in } (0,1)^2,\ g_D \equiv 0,\ D(x) = \frac{1}{0.01+100|x|^2}$$

	BJ	Dict(0.1)		Dict(0.01)		Dict(0.001)	
#levels	$k=10$	$k=10$	(%)	$k=10$	(%)	$k=10$	(%)
5	22	27		24		24	
6	24	27		24		24	
7	23	27		23		23	
8	23	28		23		23	
9	23	28	(0.05)	23	(0.7)	23	(6.1)
10	24	29	(0.01)	24	(0.2)	24	(2.3)

(*relative inverses on finest level*)

CG–iterations: $\mathbb{Q}_k^-$ elements. (Scaled) 1-Inverse does not converge.

Diffusion - Iterations 2D

$$-\nabla \cdot (D \nabla u) = 1 \text{ in } (0,1)^2,\ g_D \equiv 0,\ D(x) = \frac{1}{\alpha + \beta |x|^2}$$

	$\alpha = 0.001, \beta = 100$					$\alpha = 0.001, \beta = 10000$				
#levels	BJ	Dict(0.1)	(%)	Dict(0.05)	(%)	BJ	Dict(0.1)	(%)	Dict(0.05)	(%)
5	31	40		36		31	39		29	
6	27	33		29		30	38		31	
7	26	32		28		36	44		40	
8	26	30		27		39	47		36	
9	24	30	(0.08)	28	(0.33)	32	39	(0.08)	33	(0.35)
10	26	31	(0.02)	29	(0.09)	30	38	(0.02)	33	(0.1)

(*relative inverses on finest level*)

CG–iterations: $\mathbb{Q}_{10}^{-}$ elements. (Scaled) 1-Inverse does not converge.

Separable Operators

Let $\Omega \subset \mathbb{R}^2$ be a domain, $H = H_1 \otimes H_2$ be a tensor product of Hilbert spaces and $\mathcal{L} : H \to H$ a separable, differential operator,

$$\mathcal{L}u = \left(\mathcal{L}_1 \otimes \mathrm{id} + \mathrm{id} \otimes \mathcal{L}_2\right) u_1 \otimes u_2.$$

Let a, a_1, a_2 denote bilinear forms associated to $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2$,

$$a(u, v) = a_1(u_1, v_1)\,(u_2, v_2)_{L^2} + (u_1, v_1)_{L^2}\, a_2(u_2, v_2).$$

A FEM discretization leads to a sum of Kronecker products of one dimensional stiffness matrices

$$A = L^{(1)} \otimes M^{(2)} + M^{(1)} \otimes L^{(2)}.$$

Fast Diagonalization Method (FDM)[5]

For convenience, we assume $\mathcal{L}$ to be symmetric.
Compute one dimensional generalized eigenvalue problems

$$Q^T L^{(1)} Q = \Lambda^{(1)}, \qquad Q^T M^{(1)} Q = I,$$
$$P^T L^{(2)} P = \Lambda^{(2)}, \qquad P^T M^{(2)} P = I,$$

A tensor-product basis transformation of A into the eigenvector spaces results in a diagonal matrix,

$$(Q \otimes P)^T A (Q \otimes P) = \Lambda^{(1)} \otimes I + I \otimes \Lambda^{(2)}.$$

Therefore it is sufficient to solve small generalized eigenvalue problems to obtain the inverse of A,

$$A^{-1} = Q \otimes P \left[\Lambda^{(1)} \otimes I + I \otimes \Lambda^{(2)}\right]^{-1} Q^T \otimes P^T.$$

[1] R.E. Lynch, J.R. Rice, and D.H. Thomas. "Tensor product analysis of partial difference equations". In: *Bulletin of the American Mathematical Society* 70.3 (1964), pp. 378–384

FDM - Reduced Complexities

Let d denote the spatial dimension and n the number of degrees of freedom in one direction per element.

inverting	we requires $\mathcal{O}(dn^3)$ operations to invert A
storage	d matrices of eigenvectors and d vectors of eigenvalues
vmult	we exploit the Kronecker decompositions of the basis transformation matrices for sum-factorized matrix-vector multiplications

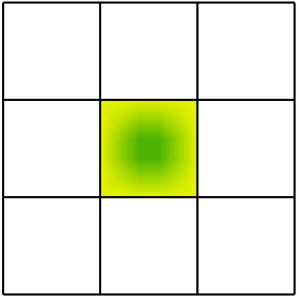
	standard	*tensor-product*
inverting	$\mathcal{O}(n^{3d})$	$\mathcal{O}(dn^3)$
storage	$\mathcal{O}(n^{2d})$	$\mathcal{O}(dn^2)$
vmult	$\mathcal{O}(n^{2d})$	$\mathcal{O}(dn^{d+1})$

Table: Complexity comparison to standard inverting

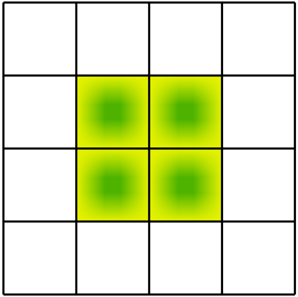
Schwarz Smoothers

Local Homogeneous Poisson Problem

Solve $-\Delta u = 1$ in Ω_S, $u = 0$ on $\partial\Omega_S$, using a local SIPG discretization.



(a) *cell* patch Ω_S



(b) regular *vertex* patch Ω_S

Additive Cell-Patch

Levels	Convergence steps			
2D	$k = 3$	$k = 4$	$k = 7$	$k = 10$
5	15	15	19	21
6	15	14	18	20
7	15	14	18	20
8	14	14	17	19
9	14	13	17	19
10	14	13	17	19
3D	$k = 3$	$k = 4$	$k = 7$	$k = 10$
2	16	16	22	25
3	18	17	22	25
4	18	17	23	25
5	18	17	22	25
6	17	17	22	24

Table: $\mathbb{Q}_k^-$ elements. Relaxation $\omega = 0.7$.

Additive Vertex-Patch

Levels	Convergence steps			
2D	$k = 3$	$k = 4$	$k = 7$	$k = 10$
5	18	18	21	23
6	17	18	21	23
7	17	18	20	23
8	17	18	20	22
9	17	18	19	21
10	17	18	19	20
3D	$k = 3$	$k = 4$	$k = 7$	$k = 10$
2	21	21	21	21
3	26	26	28	29
4	28	30	32	36
5	28	32	37	39
6	28	32	39	42

Table: $\mathbb{Q}_k^-$ elements. Relaxation $\omega = 0.25$ (0.15 in 3D).

Subdomain Coloring

Graph Coloring Algorithm [6]

First partition the mesh into zones, then colorize with respect to a given conflict type (DSATUR algorithm) and finally gather colors among zones.[7]

cell-patch	red-black coloring, 2 colors in 2D & 3D
vertex-patch (FC)	shared face defines a conflict, minimal number of colors is 8 in 2D and 16 in 3D
vertex-patch (CC)	shared cell defines a conflict, minimal number of colors is 4 in 2D and 8 in 3D

Table: optimal numbers of colors on a uniformly refined unit square/cube

[2] B. Turcksin, M. Kronbichler, and W. Bangerth. “*WorkStream* – a design pattern for multicore-enabled finite element computations”. In: *ACM Transactions on Mathematical Software* 43.1 (2016), pp. 2/1–2/29

[3] Algorithm is tuned to schedule tasks equally, not to minimize the number of colors.

Multiplicative Cell-Patch

Levels	Convergence steps				Colors
2D	$k = 3$	$k = 4$	$k = 7$	$k = 10$	
5	7.5	7.8	10.4	11.4	2
6	7.5	7.7	10.3	11.3	2
7	7.5	7.6	10.3	11.2	2
8	7.4	7.5	10.2	10.8	2
9	7.3	7.4	9.9	10.6	2
10	7.2	7.3	9.8	10.5	2
3D	$k = 3$	$k = 4$	$k = 7$	$k = 10$	
2	8.1	8.4	11.8	13.5	2
3	8.3	8.5	11.9	13.5	2
4	8.4	8.8	11.9	13.3	2
5	8.4	8.9	11.8	13.3	2
6	8.4	8.8	11.7	12.8	2

Table: $\mathbb{Q}_k^-$ elements. Relaxation $\omega = 1.0$ (1.1 in 3D).

Multiplicative Vertex-Patch (FC)

Levels	Convergence steps				Colors
2D	$k = 3$	$k = 4$	$k = 7$	$k = 10$	
5	2.7	3.0	2.0	2.0	16
6	2.7	3.0	2.0	2.0	17
7	2.8	3.0	2.0	2.0	17
8	2.6	2.8	2.1	2.0	17
9	2.5	2.6	2.1	2.0	17
10	2.5	2.5	2.0	2.0	17
3D	$k = 3$	$k = 4$	$k = 7$	$k = 10$	
2	2.0	2.0	2.0	1.5	1
3	2.1	2.2	2.0	2.0	19
4	2.2	2.3	2.0	2.0	36
5	2.3	2.4	2.0	2.0	50
6	2.2	2.4	2.0	2.0	52

Table: $\mathbb{Q}_k^-$ elements. Relaxation $\omega = 1.0$.

Multiplicative Vertex-Patch (CC)

Levels	Convergence steps				Colors
2D	$k = 3$	$k = 4$	$k = 7$	$k = 10$	
5	2.0	2.2	2.0	2.0	6
6	2.0	2.2	2.0	2.0	6
7	2.0	2.2	2.0	2.0	6
8	2.0	2.1	2.0	2.0	6
9	2.0	2.1	2.0	2.0	6
10	2.0	2.1	2.0	2.0	6
3D	$k = 3$	$k = 4$	$k = 7$	$k = 10$	
2	2.0	2.0	2.0	2.0	1
3	2.0	2.1	2.0	2.0	14
4	2.0	2.1	2.0	2.0	16
5	2.0	2.1	2.0	2.0	16
6	2.0	2.1	2.0	2.0	16

Table: $\mathbb{Q}_k^-$ elements. Relaxation $\omega = 1.0$.

Convergence Comparison 2D

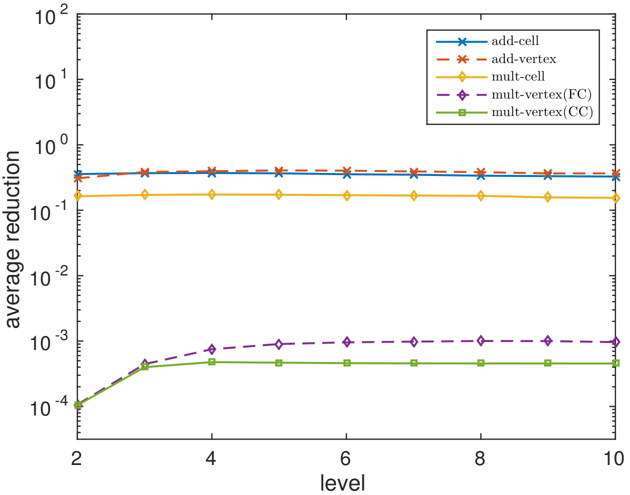


Figure: Average reduction for GMRES and $\mathbb{Q}_7^-$ elements

Convergence Comparison 3D

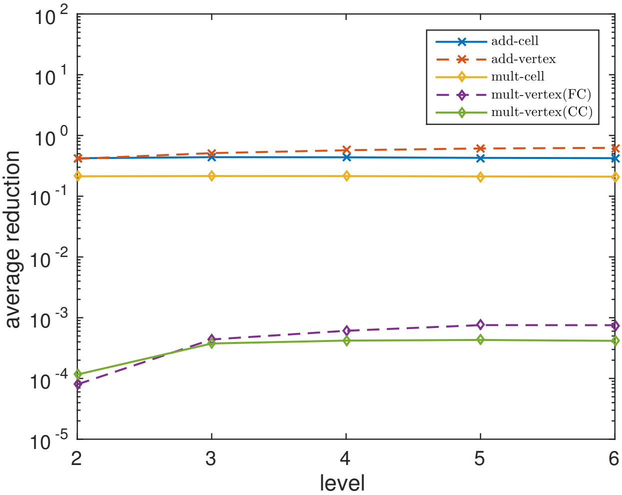


Figure: Average reduction for GMRES and $\mathbb{Q}_7^-$ elements

Summary

Results

- Local Schwarz smoothers applicable for inf-sup stable conforming elements
- Comparable results to Raviart-Thomas elements
- $\mathbb{Q}_{k+1}/\mathbb{P}_k^-$ elements perform much better than $\mathbb{Q}_{k+1}/\mathbb{Q}_k$ elements
- Analysis requires $\tau_{gd} \lesssim \nu$; sharpness confirmed by numerical results (at least for continuous pressure)
- Positive effect of stabilization especially for $\mathbb{Q}_{k+1}/\mathbb{Q}_k$ elements
- Dictionary approach/FDM shows promising results for different problems

Outlook/Challenges:

- Consider also convection dominated problems (Oseen, Navier-Stokes)
- Lift the restriction $\tau_{gd} \lesssim \nu$?

Thank you for your attention!

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