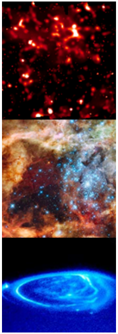
Application to Coupled Flow Problems



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The Full Set of Equations

Velocity and Pressure

$$\partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p + 2\boldsymbol{\omega} \times \boldsymbol{u} = \boldsymbol{f}_u - \beta\theta\boldsymbol{g} + (\nabla \times \boldsymbol{b}) \times \boldsymbol{b},$$
$$\nabla \cdot \boldsymbol{u} = 0$$

Magnetic Field

$$\partial_t \boldsymbol{b} + \lambda \nabla \times (\nabla \times \boldsymbol{b}) - \nabla \times (\boldsymbol{u} \times \boldsymbol{b}) = \boldsymbol{f}_b,$$
$$\nabla \cdot \boldsymbol{b} = 0$$

Temperature

$$\partial_t \theta - \alpha \Delta\theta + (\boldsymbol{u} \cdot \nabla)\theta = f_\theta$$

Local Projection Stabilization

Idea

- Separate discrete function spaces into small and large scales
- Add stabilization terms only on small scales.

Notations and prerequisites

- Family of shape-regular macro decompositions $\{\mathcal{M}_h\}$
- Let $D_M \subset [L^\infty(M)]^d$ denote a FE space on $M \in \mathcal{M}_h$.
- For each $M \in \mathcal{M}_h$, let $\pi_M \colon [L^2(M)]^d \to D_M$ be the orthogonal L^2-projection.
- $\kappa_M = Id - \pi_M$ fluctuation operator
- Averaged streamline direction $\boldsymbol{u}_M \in \mathbb{R}^d$:
  $|\boldsymbol{u}_M| \leq C\|\boldsymbol{u}\|_{L^\infty(M)}, \ \|\boldsymbol{u} - \boldsymbol{u}_M\|_{L^\infty(M)} \leq Ch_M|\boldsymbol{u}|_{W^{1,\infty}(M)}$

Assumptions

Assumption - Approximation

It holds for all $w \in W^{l,2}(M)$, $M \in \mathcal{M}_h$ and $l \leq s \leq k$

$$\|\kappa_M \boldsymbol{w}\|_{L^2(M)} \leq C h_M^l \|\boldsymbol{w}\|_{W^{l,2}(M)}$$

Assumption - Inf-Sup Stability

Consider FE spaces (V_h, Q_h) satisfying a discrete inf-sup-condition:

$$\inf_{q \in Q_h \setminus \{0\}} \sup_{v \in V_h \setminus \{0\}} \frac{(\nabla \cdot v, q)}{\|\nabla v\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)}} \geq \beta > 0$$

$$\Rightarrow \quad \boldsymbol{V_h}^{div} := \{v_h \in V_h \mid (\nabla \cdot v_h, q_h) = 0 \ \ \forall q_h \in Q_h\} \neq \{0\}$$

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Rotating Frames of Reference

Navier Stokes Equations in an Inertial Frame of Reference

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad \text{in } \Omega \times (0, T)$$

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \text{in } \Omega \times (0, T)$$

$\Omega \subset \mathbb{R}^d$ bounded polyhedral domain

Navier Stokes Equations in a Rotating Frame of Reference

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nu \Delta \boldsymbol{v} + 2\boldsymbol{\omega} \times \boldsymbol{v} + \nabla \widetilde{p} = \boldsymbol{f} \qquad \text{in } \Omega \times (0, T)$$

$$\nabla \cdot \boldsymbol{v} = 0 \qquad \text{in } \Omega \times (0, T)$$

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) = -\frac{1}{2}\nabla(\boldsymbol{\omega} \times \boldsymbol{r})^2 \qquad \widetilde{p} = p - \frac{1}{2}(\boldsymbol{\omega} \times \boldsymbol{r})^2$$

Weak Formulation

Find $\boldsymbol{\mathcal{U}}_h = (\boldsymbol{u}_h, p_h) : (0, T) \to \boldsymbol{V}_h \times Q_h$, such that

$$(\partial_t \boldsymbol{u}_h, \boldsymbol{v}_h) + \mathcal{A}_G(\boldsymbol{u}_h, \boldsymbol{\mathcal{U}}_h, \boldsymbol{\mathcal{V}}_h) + (2\boldsymbol{\omega} \times \boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h)$$

for all $\boldsymbol{\mathcal{V}}_{\boldsymbol{h}} = (\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times Q_h$

where

$$\begin{aligned}
\mathcal{A}_G(\boldsymbol{w}; \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{V}}) &:= a_G(\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{V}}) + c(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) \\
a_G(\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{V}}) &:= \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (p, \nabla \cdot \boldsymbol{v}) + (q, \nabla \cdot \boldsymbol{u}) \\
c(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) &:= \frac{((\boldsymbol{w} \cdot \nabla)\boldsymbol{u}, \boldsymbol{v}) - ((\boldsymbol{w} \cdot \nabla)\boldsymbol{v}, \boldsymbol{u})}{2}
\end{aligned}$$

Stabilization Terms

- LPS Streamline upwind Petrov-Galerkin (SUPG)

$$s_u(\boldsymbol{w}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) := \sum_{M \in \mathcal{M}_h} \tau_M(\boldsymbol{w}_M)(\kappa_M((\boldsymbol{w}_M \cdot \nabla)\boldsymbol{u}_h), \kappa_M((\boldsymbol{w}_M \cdot \nabla)\boldsymbol{v}_h))_M$$

- grad-div

$$t_h(\boldsymbol{w}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) := \sum_{M \in \mathcal{M}_h} \gamma_M(\boldsymbol{w}_M)(\nabla \cdot \boldsymbol{u}_h, \nabla \cdot \boldsymbol{v}_h)_M$$

- LPS Coriolis stabilization

$$a_h(\boldsymbol{w}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) := \sum_{M \in \mathcal{M}_h} \alpha_M(\boldsymbol{w}_M)(\kappa_M(\boldsymbol{\omega}_M \times \boldsymbol{u}_h), \kappa_M(\boldsymbol{\omega}_M \times \boldsymbol{v}_h))_M$$

Convergence Result

Theorem

For a sufficiently smooth solution we obtain for $\boldsymbol{e}_h = \boldsymbol{u_h} - j_u \boldsymbol{u}$:

$$\|\boldsymbol{e}_h\|^2_{L^\infty(0,t;[L^2(\Omega)]^d)} + \int_0^t |||\boldsymbol{e}_h(\tau)|||^2_{LPS}\, d\tau \leq C \exp(C_G t) h^{2k}$$

with a Gronwall constant
$C_G(\boldsymbol{u}) = 1 + C|\boldsymbol{u}|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + Ch\|\boldsymbol{u}\|^2_{L^\infty(0,T;W^{1,\infty}(\Omega))}$

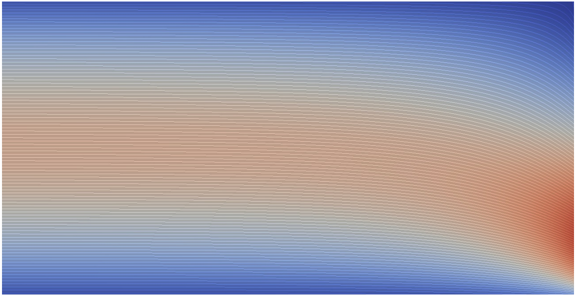
The parameters have to satisfy $(1 \leq s \leq k)$:

$$h_M \leq C \frac{\sqrt{\nu}}{\|\boldsymbol{u}_h\|_{L^\infty(M)}} \qquad \tau_M \leq \tau_0 \frac{h_M^{2(k-s)}}{|\boldsymbol{u}_M|^2}$$

$$\gamma_M = \gamma_0 \qquad \alpha_M \leq \alpha_0 \frac{h_M^{2(k-s-1)}}{\|\boldsymbol{\omega}\|^2_{L^\infty(M)}}$$

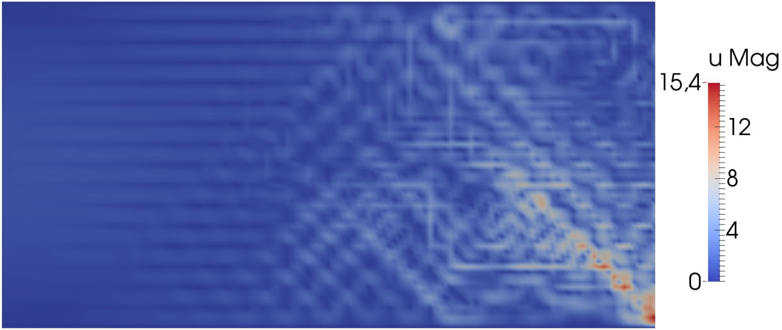
Numerical Results, Rotating Poiseuille Flow

- $\Omega = [-2, 2] \times [-1, 1]$
- $\boldsymbol{u}(x, y) = \begin{cases} (1 - y^2, 0)^T, & x = -2 \\ (0, 0)^T, & |y| = 1 \end{cases}, \quad (\nabla \boldsymbol{u} \cdot \boldsymbol{n})(x = 2, y) = 0$
- $\boldsymbol{u}_0 = 0, \quad p_0 = 0, \quad \boldsymbol{f} = 0 \quad \boldsymbol{\omega} = (0, 0, 100), \quad \nu = 10^{-3}$

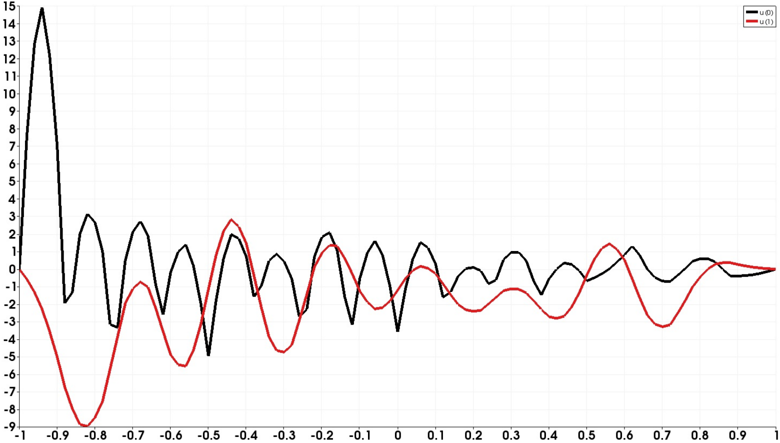


Flow for the parameters $\boldsymbol{\omega} = (0, 0, 1)$, $\nu = 10^{-1}$

Rotating Poiseuille Flow, grad-div

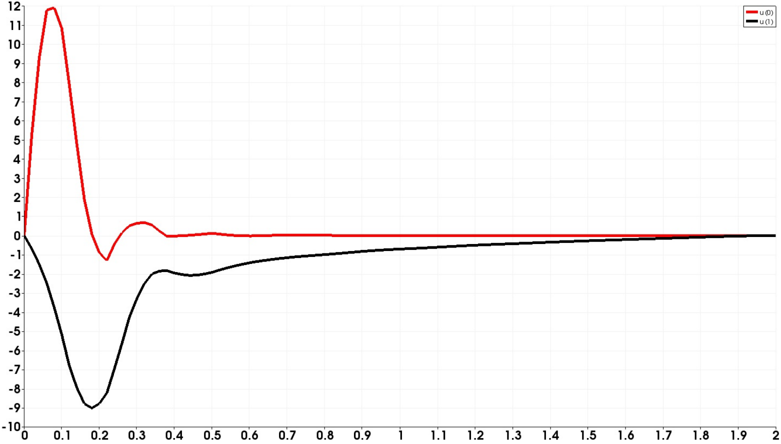


Rotating Poiseuille Flow, grad-div

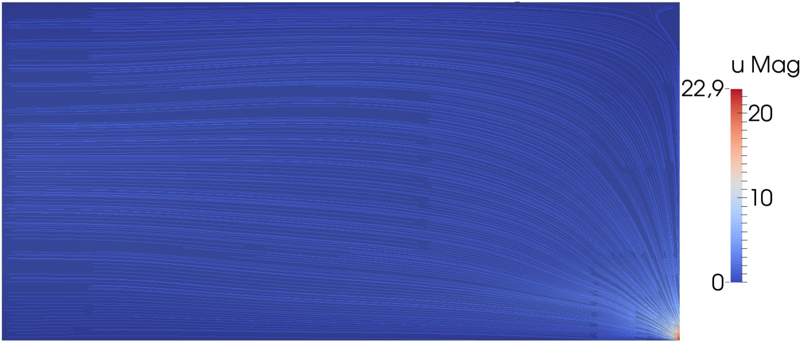


Rotating Poiseuille Flow, SUPG Coriolis

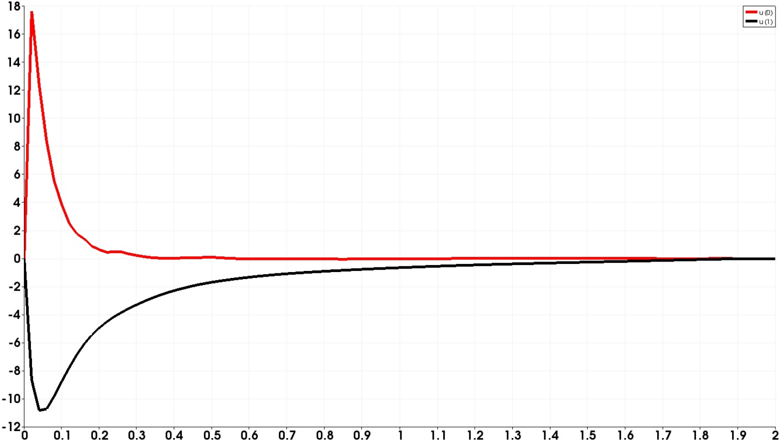
Rotating Poiseuille Flow, SUPG Coriolis



Rotating Poiseuille Flow, SUPG Coriolis Adaptive



Rotating Poiseuille Flow, SUPG Coriolis Adaptive



Taylor-Proudman, Stewartson Layer

Navier Stokes Equations in a Rotating Frame of Reference

$$\frac{\partial \boldsymbol{u}}{\partial t} + Ro(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + 2\hat{\mathbf{e}}_z \times \boldsymbol{u} = Ek\Delta\boldsymbol{u} - \nabla p$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\boldsymbol{u} = r \sin\theta \hat{\mathbf{e}}_\phi \qquad \text{at } r = r_i$$

$$\boldsymbol{u} = \mathbf{0} \qquad \text{at } r = r_o$$

$$r_i = 1/2 \qquad r_o = 3/2$$

$$Ro := \Delta\Omega/\Omega \qquad Ek := \frac{\nu}{\Omega(r_o - r_i)^2}$$

Taylor-Proudman, Stewartson Layer

Fluid structure between two rotating spheres

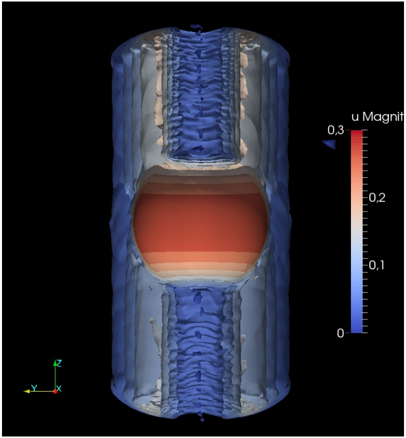


Figure: $Ek = 10^{-6}$

Taylor-Proudman, Stewartson Layer

Fluid structure between two rotating spheres

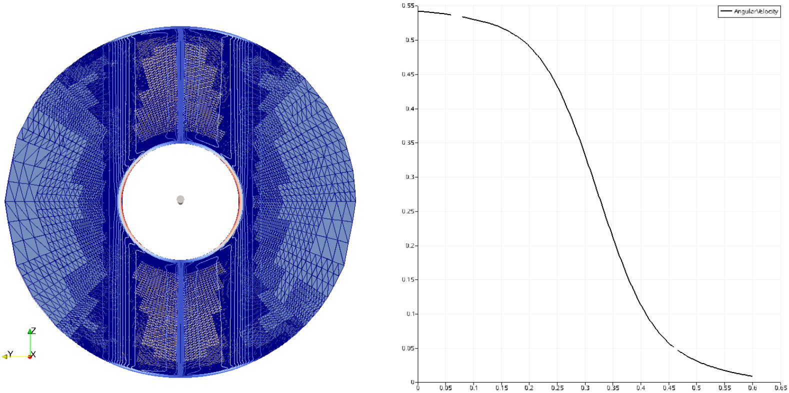


Figure: $Ek = 10^{-4} Ro = -.5$

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Incompressible Nodal-based MHD - Joint work with Benjamin Wacker

Stationary Linearized Incompressible Nodal-based MHD model

$$
\begin{aligned}
-\nu\Delta\boldsymbol{u} + (\boldsymbol{a}\cdot\nabla)\boldsymbol{u} + \nabla p - (\nabla\times\boldsymbol{b})\times\boldsymbol{d} &= \boldsymbol{f}_u,\\
\nabla\cdot\boldsymbol{u} &= 0,\\
\lambda\nabla\times(\nabla\times\boldsymbol{b}) + \nabla r - \nabla\times(\boldsymbol{u}\times\boldsymbol{d}) &= \boldsymbol{f}_b,\\
\nabla\cdot\boldsymbol{b} &= 0
\end{aligned}
$$

- $\boldsymbol{u}$ velocity field, p kinematic pressure
- $\boldsymbol{a}$ extrapolation for $\boldsymbol{u}$
- $\boldsymbol{b}$ induced magnetic field, r magnetic pseudo pressure
- $\boldsymbol{d}$ extrapolation for $\boldsymbol{b}$
- ν kinematic viscosity, λ magnetic diffusivity

Weak Formulation

Find $\mathcal{U}_h := (\boldsymbol{u}_h, \boldsymbol{b}_h, p_h, r_h) \in \boldsymbol{V}_h \times \boldsymbol{C}_h \times Q_h \times S_h$ such that

$$\mathcal{A}_{G,u}(\mathcal{U}_h, \mathcal{V}_h) + \mathcal{A}_{G,b}(\mathcal{U}_h, \mathcal{V}_h) = \langle \boldsymbol{f}_u, \boldsymbol{v} \rangle + \langle \boldsymbol{f}_b, \boldsymbol{c} \rangle,$$

for all $\mathcal{V}_h := (\boldsymbol{v}_h, \boldsymbol{c}_h, q_h, s_h) \in \boldsymbol{V}_h \times \boldsymbol{C}_h \times Q_h \times S_h$.

$$\begin{aligned}
\mathcal{A}_{G,u}(\boldsymbol{U}, \boldsymbol{V}) &= \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + \langle \boldsymbol{a} \cdot \nabla \boldsymbol{u}, \boldsymbol{v} \rangle - \langle (\nabla \times \boldsymbol{b}) \times \boldsymbol{d}, \boldsymbol{v} \rangle \\
&\quad - (p, \nabla \cdot \boldsymbol{v}) + (\nabla \cdot \boldsymbol{u}, q) \\
\mathcal{A}_{G,b}(\boldsymbol{U}, \boldsymbol{V}) &= \lambda(\nabla \times \boldsymbol{b}, \nabla \times \boldsymbol{c}) - \langle \nabla \times (\boldsymbol{u} \times \boldsymbol{d}), \boldsymbol{c} \rangle \\
&\quad + (\nabla r, \boldsymbol{c}) - (\boldsymbol{b}, \nabla s)
\end{aligned}$$

Stabilization

- LPS-SU

$$s_1(\boldsymbol{u}_h, \boldsymbol{v}_h)_M := \tau_{SU}(\kappa_M((\boldsymbol{a}_M \cdot \nabla)\boldsymbol{u}_h), \kappa_M((\boldsymbol{a}_M \cdot \nabla)\boldsymbol{v}_h))_M$$

- grad-div velocity

$$s_2(\boldsymbol{u}_h, \boldsymbol{v}_h)_M := \tau_{gd,u}(\nabla \cdot \boldsymbol{u}_h, \nabla \cdot \boldsymbol{v}_h)_M$$

- LPS-Lorentz

$$s_3(\boldsymbol{b}_h, \boldsymbol{c}_h)_M := \tau_{Lor}(\kappa_M((\nabla \times \boldsymbol{b}_h) \times \boldsymbol{d}_M), \kappa_M((\nabla \times \boldsymbol{c}_h) \times \boldsymbol{d}_M))_M$$

- LPS-Induction

$$s_4(\boldsymbol{u}_h, \boldsymbol{v}_h)_M := \tau_{Ind}(\kappa_M(\nabla \times (\boldsymbol{u}_h \times \boldsymbol{d}_M)), \kappa_M(\nabla \times (\boldsymbol{v}_h \times \boldsymbol{d}_M)))_M$$

- PSPG

$$s_5(r_h, s_h)_M := \tau_{PSPG}(\nabla r_h, \nabla s_h)_M$$

- grad-div magnetic

$$s_6(\boldsymbol{b}_h, \boldsymbol{c}_h)_M := \tau_{gd,b}(\nabla \cdot \boldsymbol{b}_h, \nabla \cdot \boldsymbol{c}_h)_M$$

Convergence Result

Theorem

For sufficiently smooth solutions we obtain:

$$\|\boldsymbol{U}_h - \boldsymbol{U}\|_{\mathrm{G}}^2 + \|\boldsymbol{U}_h - \boldsymbol{U}\|_{\mathrm{LPS}}^2 \leq C \sum_M h_M^{2k} \left(|\boldsymbol{u}|_{k+1,\omega_M}^2 + |\boldsymbol{b}|_{k+1,\omega_M}^2 + |p|_{k,\omega_M}^2 \right)$$

with a parameter choice according to

$$h_M \leq C \min \left\{ \frac{\sqrt{\nu}}{\|\boldsymbol{a}\|_{\infty,M}}, \frac{\sqrt{\lambda}}{\|\boldsymbol{d}\|_{\infty_M}}, \frac{\sqrt{\nu}}{\|\boldsymbol{d}\|_{\infty,M} + \|\nabla \boldsymbol{d}\|_{\infty,M}} \right\}$$

$$\tau_{SU} \leq C \frac{h_M^{2(k-s)}}{|\boldsymbol{a}_M|^2} \qquad \tau_{gd,u} \sim \gamma_0 \qquad \tau_{Lor}, \tau_{Ind} \leq C \frac{h_M^{2(k-s)}}{|\boldsymbol{d}_M|^2}$$

$$\tau_{PSPG} \sim \frac{L_{0,m}^2}{\lambda} \qquad \tau_{gd,b} \sim \frac{h_M^2 \lambda}{L_{0,m}^2}$$

Analytical 2D Problem

Consider on $\Omega = (-1,1) \times (-1,1)$ for $t \in [0, 0.1]$ the solution:

$$\begin{aligned} \boldsymbol{u}(t,x,y) &:= \boldsymbol{b}(t,x,y) \\ &:= \exp\left(\frac{t}{25}\right)\left(x^4 \sin\left(\frac{\pi t}{10}\right), -4x^3 y \sin\left(\frac{\pi t}{10}\right)\right)^T, \\ p(x,y) &:= x^2 + y^2, \qquad r(x,y) := 0 \end{aligned}$$

Boundary conditions: $\quad \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}_b \qquad \mathbf{n} \times \mathbf{b} = \mathbf{n} \times \mathbf{b_D}$

Parameters:

$$\Delta t = 10^{-4}, \qquad \nu = 10^{-6}, \qquad \lambda = 10^{-6}$$

$$\tau_2 = 1, \qquad \tau_5 = \frac{h^2 \lambda}{L_0^2}, \qquad \tau_6 = \frac{L_0^2}{\lambda}, \qquad L_0 = 2$$

Ansatz spaces: $V_h / Q_h = C_h / S_h = [\mathbb{Q}_2]^2 / \mathbb{Q}_1$

Convergence Results

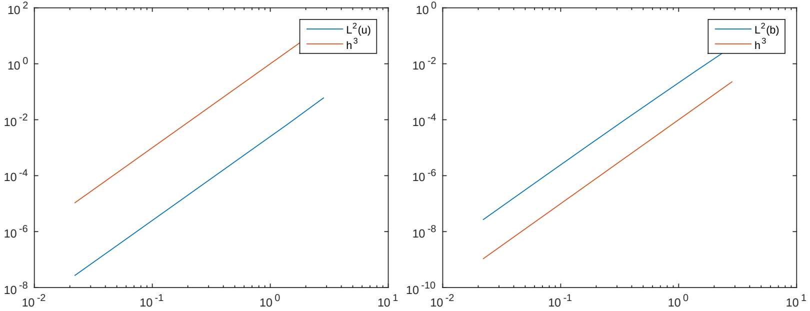

Figure: $L^2(\Omega)$ Errors for the 2D-time-dependent analytical problem

Convergence Results

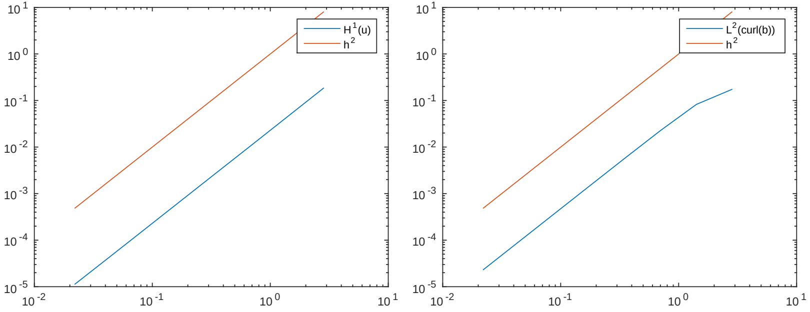

Figure: Errors for the 2D-time-dependent analytical problem

Convergence Results

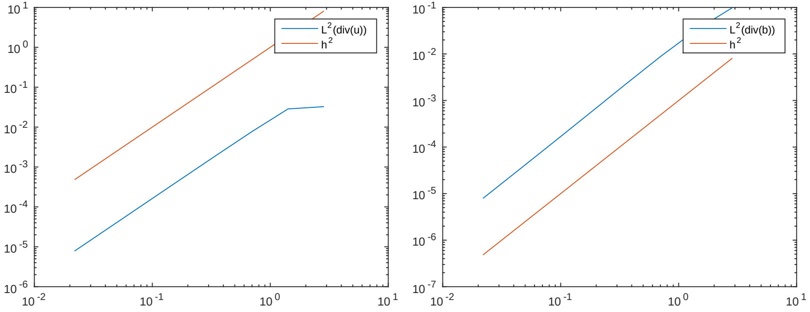

Figure: Divergence Errors for the 2D-time-dependent analytical problem

Convergence Results

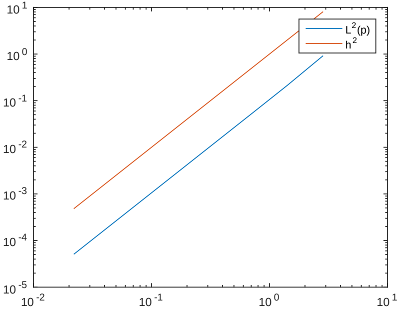

Figure: Pressure Errors for the 2D-time-dependent analytical problem

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Non-Isothermal Flow - Joint work with Helene Dallmann

Oberbeck-Boussinesq-Model

$$\begin{aligned} \partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f}_u - \beta \theta \boldsymbol{g} \\ \nabla \cdot \boldsymbol{u} &= 0 \\ \partial_t \theta - \alpha \Delta \theta + (\boldsymbol{u} \cdot \nabla)\theta &= f_\theta \end{aligned}$$

- velocity $\boldsymbol{u}$, kinematic pressure p, temperature θ
- thermal diffusivity α, thermal expansion coefficient β, gravity $\boldsymbol{g}$
- small temperature differences $\Rightarrow$ density $\varrho \approx const.$

Weak Formulation

Find $(\boldsymbol{\mathcal{U}}_h, \theta_h) = (\boldsymbol{u}_h, p_h, \theta_h) : (0, T) \to \boldsymbol{V}_h^{div} \times Q_h \times \Theta_h$ such that

$$(\partial_t \boldsymbol{u}_h, \boldsymbol{v}_h) + \mathcal{A}_{G,u}(\boldsymbol{u}_h, \boldsymbol{\mathcal{U}}_h, \boldsymbol{\mathcal{V}}_h) = (f, \boldsymbol{v}_h) - (\beta \theta_h g, \boldsymbol{v}_h)$$
$$(\partial_t \theta_h, \psi_h) + \mathcal{A}_{G,\theta}(\boldsymbol{u}_h, \theta_h, \psi_h) = (f_\theta, \psi_h)$$

for all $(\boldsymbol{V}_h, \psi_h) = (\boldsymbol{v}_h, q_h, \psi_h) \in \boldsymbol{V}_h \times Q_h \times \Theta_h$

where

$$\mathcal{A}_{G,u}(\boldsymbol{w}; \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{V}}) := a_G(\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{V}}) + c(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})$$
$$a_G(\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{V}}) := \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (p, \nabla \cdot \boldsymbol{v}) + (q, \nabla \cdot \boldsymbol{u})$$
$$\mathcal{A}_{G,\theta}(\boldsymbol{u}, \theta, \psi) := \alpha(\nabla \theta, \nabla \psi) + c(\boldsymbol{u}; \theta, \psi)$$
$$c(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) := \frac{((\boldsymbol{w} \cdot \nabla)\boldsymbol{u}, \boldsymbol{v}) - ((\boldsymbol{w} \cdot \nabla)\boldsymbol{v}, \boldsymbol{u})}{2}$$

Stabilization terms

- LPS-SUPG for the velocity

$$s_u(\boldsymbol{u}_h; \boldsymbol{w}_h, \boldsymbol{v}_h) := \sum_{M \in \mathcal{M}_h} \tau_M^u(\boldsymbol{u}_h)(\kappa_M^u(\boldsymbol{u}_M \cdot \nabla \boldsymbol{w}_h), \kappa_M^u(\boldsymbol{u}_M \cdot \nabla \boldsymbol{v}_h))_M$$

- grad-div

$$t_h(\boldsymbol{u}_h; \theta_h, \psi_h) := \sum_{M \in \mathcal{M}_h} \gamma_M(\boldsymbol{u}_h)(\nabla \cdot \boldsymbol{w}_h, \nabla \cdot \boldsymbol{v}_h)_M$$

- LPS-SUPG for the temperature

$$s_\theta(\boldsymbol{u}_h; \theta_h, \psi_h) := \sum_{M \in \mathcal{M}_h} \tau_M^\theta(\boldsymbol{u}_M)(\kappa_M^\theta(\boldsymbol{u}_M \cdot \nabla \theta_h), \kappa_M^\theta(\boldsymbol{u}_M \cdot \nabla \psi_h))_M$$

Convergence Result

Theorem

For sufficiently smooth solutions we obtain for $\boldsymbol{e}_h = \boldsymbol{u} - \boldsymbol{u}_h$ and $e_\theta = \theta - \theta_h$:

$$\|\boldsymbol{e}_h\|^2_{L^\infty(0,t;L^2(\Omega))} + \int_0^t |||(\boldsymbol{e}_h(\tau), 0)|||^2_{LPS}\, d\tau + \|e_\theta\|^2_{L^\infty(0,t;L^2(\Omega))} + \int_0^t |[e_\theta(\tau)]|^2_{LPS}\, d\tau \leq \mathcal{O}(h^{2k})$$

with a parameter choice according to

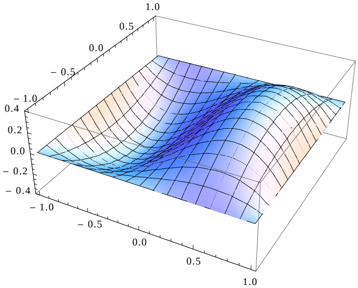
$$h_M \leq C \frac{\sqrt{\nu}}{\|\boldsymbol{u}\|_{L^\infty(M)}} \qquad h_M \leq C \frac{\sqrt{\alpha}}{\|\boldsymbol{u}\|_{L^\infty(M)}}$$

$$\tau_M \leq \tau_0 \frac{h_M^{2(k-s)}}{|\mathbf{u}_M|^2} \qquad \tau_M^\theta \leq \tau_{0,\theta} \frac{h_M^{2(k-s_\theta)}}{|\mathbf{u}_M|^2}$$

$$\gamma_M = \gamma_0$$

Bubble Enriched Ansatz Spaces

Enrich the tensor product-polynomial space $\mathbb{Q}_k$ to obtain



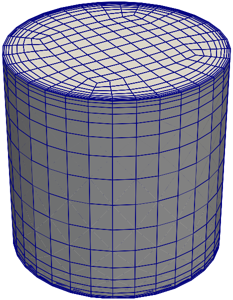
$$\mathbb{Q}_k^+(\hat{T}) := \mathbb{Q}_k(\hat{T}) + \psi \cdot \mathrm{span}\{\hat{x}_i^{k-1},\ i = 1, \ldots, d\}$$

with bubble functions
$\psi(\hat{\boldsymbol{x}}) := \prod_{i=1}^d (1 - \hat{x}_i^2)$.

Used spaces

- Taylor-Hood elements ($\mathbb{Q}_2/\mathbb{Q}_1$) for velocity and pressure
- $\mathbb{Q}_2$ or $\mathbb{Q}_2^+$ for the temperature
- $\mathbb{Q}_1$ for the projection spaces

Rayleigh–Bénard Convection



$$R = .5 \quad H = 1 \quad \beta = 1$$

$$\nu = Pr^{-0.5} \cdot Ra^{0.5}$$

$$\alpha = Ra^{-0.5} \cdot Pr^{-0.5}$$

Flow driven by temperature difference

Dirichlet BC's for bottom and top plate:

$$\theta_{bottom} = 0.5, \quad \theta_{top} = -0.5,$$

and isolating hull:

$$\boldsymbol{n} \cdot \nabla\theta|_{r=0.5} = 0$$

No-slip boundary conditions for the velocity:

$$\boldsymbol{u} = \boldsymbol{0}|_{\partial\Omega}$$

Isosurfaces of the Temperature, $Pr = 0.786$

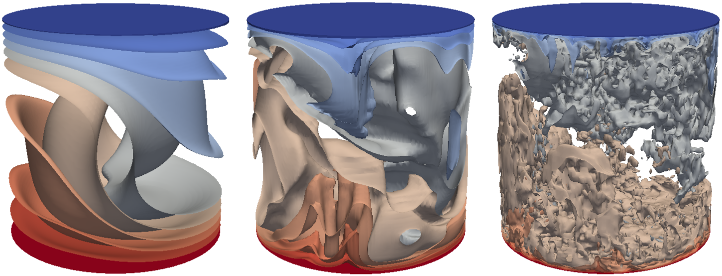


Figure: Isosurfaces of the Temperature, $T = 1000$, $Pr = 0.786$, $Ra = 10^5$, $Ra = 10^7$, $Ra = 10^9$, $N = 10 \cdot 16^3$, $\gamma_M = 0.1$

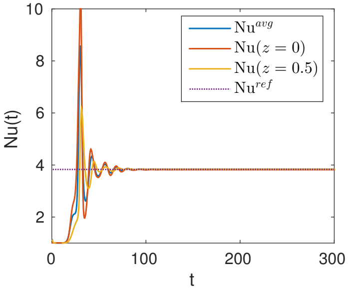
Benchmark quantity - Nusselt number

- Heat flux: bottom (warm) $\Rightarrow$ top (cold)
  $q_z := u_z\theta - \alpha\partial_z\theta$
- Nusselt number

$$Nu(z_0, t) := \frac{L}{\alpha A \Delta\theta} \langle q_z \rangle_{z=z_0}(z_0, t)$$

  measure for $\frac{\text{convective heat transfer}}{\text{conductive heat transfer}}$
- $Nu(z_0, t)$ independent of z_0 for a stationary temperature field.

Time-development



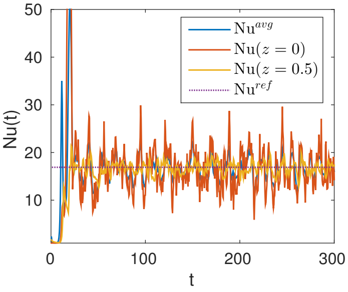


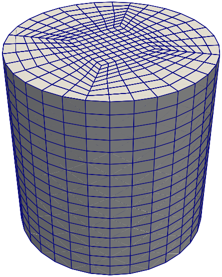
Figure: Time-development of Nu for $t \in [0, 300]$, $\gamma_M = 0.1$ $Pr = 0.786$, $Ra = 10^5$ und $Ra = 10^7$ $(N = 10 \cdot 8^3)$

Dependence on Ra ($N = 10 \cdot 8^3$ cells)

Ra	γ_M	Nu^{avg}	σ	Nu^{ref} [DNS]
10^5	0	3.8396	0.0356	3.83
	1	3.8364	0.0307	
	0.1	**3.8372**	**0.0303**	
10^6	0	8.6457	0.3378	8.6
	1	8.5148	0.0542	
	0.1	**8.6475**	**0.0190**	
10^7	0	16.4143	1.8302	16.9
	1	16.7361	0.1569	
	0.1	**16.8767**	**0.1068**	
10^8	0	37.7301	29.4731	31.9
	1	30.7236	0.7044	
	0.1	**31.2902**	**0.6957**	

Isotropic Mesh, $Ra = 10^9$ ($N = 10 \cdot 8^3$)

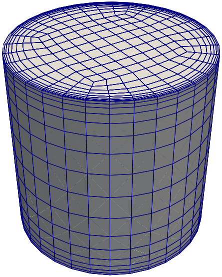
γ_M	τ_M^u	τ_L^θ	$Nu_{Id,th}^{avg}$	$\sigma_{Id,th}$	$Nu_{Id,bb}^{avg}$	$\sigma_{Id,bb}$	Nu^{ref}
0.01	0	0	41.4584	40.1989	47.5335	23.4029	63.1
0.01	hu1	0	38.7093	43.0326	44.2998	24.7851	
0.01	0	hu1	37.6081	10.8360	**54.2603**	**16.5349**	
0.01	hu1	hu1	37.0516	10.3065	49.1255	12.9235	



th: $(\mathbb{Q}_2/\mathbb{Q}_1) \wedge \mathbb{Q}_1 \wedge (\mathbb{Q}_2/\mathbb{Q}_1)$,
bb: $(\mathbb{Q}_2^+/\mathbb{Q}_1) \wedge \mathbb{Q}_1 \wedge (\mathbb{Q}_2^+/\mathbb{Q}_1)$
hu1: $\tau_{M/L}^{u/\theta} = \frac{1}{2} h / \|\boldsymbol{u}_h\|_{\infty, M/L}$

Anisotropic mesh, $Ra = 10^9$ ($N = 10 \cdot 8^3$) cells

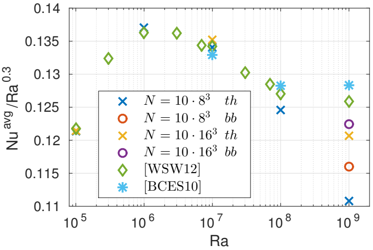
γ_M	τ_M^u	τ_L^θ	Nu_{th}^{avg}	σ_{th}	Nu_{bb}^{avg}	σ_{bb}	Nu^{ref}
0	0	0	118.7932	137.5588			63.1
0.01	0	0	**55.5231**	**1.3464**	58.1419	1.4833	
0.01	hu1	0	53.8371	1.4130	**58.2691**	**1.4702**	
0.01	0	hu1	52.4530	3.4847	56.5274	3.0578	
0.01	hu1	hu1	51.8141	3.4344	54.0410	3.3333	



th: $(\mathbb{Q}_2/\mathbb{Q}_1) \wedge \mathbb{Q}_1 \wedge (\mathbb{Q}_2/\mathbb{Q}_1)$,
bb: $(\mathbb{Q}_2^+/\mathbb{Q}_1) \wedge \mathbb{Q}_1 \wedge (\mathbb{Q}_2^+/\mathbb{Q}_1)$
hu1: $\tau_{M/L}^{u/\theta} = \frac{1}{2} h / \|\boldsymbol{u}_h\|_{\infty,M/L}$

Vary Discretization

	#cells	#DoF(u)	#DoF(p)	#DoF(θ)	$Nu^{avg}_{\gamma_M=0.01}$	$\sigma_{\gamma_M=0.01}$
th	$10 \cdot 8^3$	129,987	5,729	43,329	55.5231	1.3464
bb	$10 \cdot 8^3$	176,067	5,729	58,689	58.1419	1.4833
th	$10 \cdot 16^3$	1,011,075	43,329	337,025	60.4889	1.1574
bb	$10 \cdot 16^3$	1,379,715	43,329	459,905	61.3628	0.4668



Summary

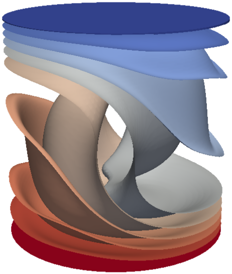
Analytical and numerical framework for LPS stabilized model of incompressible flow with

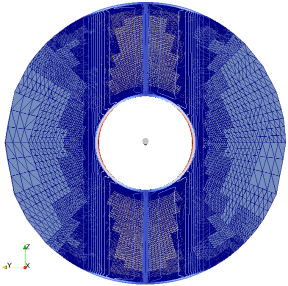
- rotating frames of references
- temperature coupling
- magnetohydrodynamic coupling

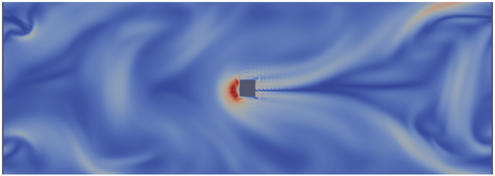
Next step: Combine all of these!

References

- Daniel Arndt, Helene Dallmann, and Gert Lube, *Local Projection FEM Stabilization for the Time-Dependent incompressible Navier-Stokes Problem*, Numerical Methods for Partial Differential Equations **31** (2015), no. 4, 1224–1250.
- Daniel Arndt and Gert Lube, *FEM with Local Projection Stabilization for Incompressible Flows in Rotating Frames*, Preprint.
- Helene Dallmann and Daniel Arndt, *Stabilized Finite Element Methods for the Oberbeck-Boussinesq Model*, In Preparation.
- Helene Dallmann, *Finite Element Methods with Local Projection Stabilization for Thermally Coupled Incompressible Flow*, Ph.D. thesis, Georg-August-Universität Göttingen, 2015.
- Benjamin Wacker, Daniel Arndt, and Gert Lube, *Nodal-Based Finite Element Methods with Local Projection Stabilization for Linearized Incompressible Magnetohydrodynamics*, CMAME (2015).







Thanks for your attention!