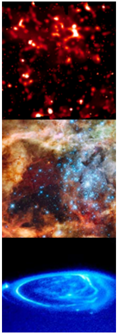
Projection Methods for Rotating Flow



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Computational Modeling of Turbulent and Complex Flows with Applications I

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Overview

Overview

Setting

Navier Stokes Equations in an Inertial Frame of Reference

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega \times (0, T]$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega \times (0, T]$$

$\Omega \subset \mathbb{R}^d$ bounded polyhedral domain

Navier Stokes Equations in a Rotating Frame of Reference

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \nu \Delta \mathbf{v} + 2\omega \times \mathbf{v} + \nabla \tilde{p} = \mathbf{f} \qquad \text{in } \Omega \times (0, T]$$

$$\nabla \cdot \mathbf{v} = 0 \qquad \text{in } \Omega \times (0, T]$$

$$\omega \times (\omega \times \mathbf{r}) = -\frac{1}{2}\nabla(\omega \times \mathbf{r})^2 \qquad \tilde{p} = p - \frac{1}{2}(\omega \times \mathbf{r})^2$$

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Overview

Pressure-Correction Scheme, Diffusion step

Find $\widetilde{\mathbf{u}}^{k+1}$ such that

$$
\begin{aligned}
&\frac{1}{2\Delta t}(3\widetilde{\mathbf{u}}^{k+1} - 4\mathbf{u}^k + \mathbf{u}^{k-1}) \\
&+ (\mathbf{u}^* \cdot \nabla \widetilde{\mathbf{u}}^{k+1} + \frac{1}{2}(\nabla \cdot \mathbf{u}^*)\widetilde{\mathbf{u}}^{k+1}) \\
&- \nu \nabla^2 \widetilde{\mathbf{u}}^{k+1} + 2\boldsymbol{\omega} \times \widetilde{\mathbf{u}}^{k+1} + \nabla p^k = \mathbf{f}(t^{k+1}) \text{ in } \Omega
\end{aligned}
$$

and satisfying

$$\widetilde{\mathbf{u}}^{k+1} = 0 \text{ on } \partial\Omega.$$

Pressure-Correction Scheme, Projection step

Find $(\mathbf{u}^{k+1}, p^{k+1})$ such that

$$\frac{1}{2\Delta t}(3\mathbf{u}^{k+1} - 3\widetilde{\mathbf{u}}^{k+1}) + \nabla\phi^{k+1} = 0 \text{ in } \Omega$$
$$\nabla \cdot \mathbf{u}^{k+1} = 0, \qquad \mathbf{u}^{k+1} \cdot \mathbf{n}|_{\partial\Omega} = 0 \text{ in } \Omega$$

and satisfying

$$\mathbf{u}^{k+1} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega.$$

The updated pressure is then given by

$$p^{k+1} = \phi^{k+1} + p^k - \alpha\nu\nabla \cdot \widetilde{\mathbf{u}}^{k+1}$$

where $\alpha \in \{0, 1\}$.

Standard Pressure-Correction Scheme, Error Estimates

Theorem (Rates of convergence for $\alpha = 0$)

Provided $(\mathbf{u}, p)$ *is sufficiently smooth,* $(\mathbf{u}_{\Delta t}, \widetilde{\mathbf{u}}_{\Delta t}, p_{\Delta t})$ *satisfies*

$$\|\mathbf{u} - \mathbf{u}_{\Delta t}\|_{l^2([L^2(\Omega)]^d)} + \|\mathbf{u} - \widetilde{\mathbf{u}}_{\Delta t}\|_{l^2([L^2(\Omega)]^d)} \leq C\Delta t^2$$
$$\|p - p_{\Delta t}\|_{l^\infty(L^2(\Omega))} + \|\mathbf{u} - \widetilde{\mathbf{u}}_{\Delta t}\|_{l^\infty([H^1(\Omega)]^d)} \leq C\Delta t$$

Proof.

Similar to the proof by Guermond and Shen in [GS04] □

Remark

- $\nabla p^{k+1} \cdot \mathbf{n}|_{\partial\Omega} = \ldots = \nabla p^0 \cdot \mathbf{n}|_{\partial\Omega}$
  non-physical Neumann boundary condition leads to numerical boundary layer, reducing the order of convergence
- the splitting-error is of size $\mathcal{O}(\Delta t^2)$

Rotational Pressure-Correction Scheme, Error Estimates

Theorem (Rates of convergence for $\alpha = 1$)

Provided $(\mathbf{u}, p)$ *is sufficiently smooth,* $(\mathbf{u}_{\Delta t}, \widetilde{\mathbf{u}}_{\Delta t}, p_{\Delta t})$ *satisfies*

$$\|\mathbf{u} - \mathbf{u}_{\Delta t}\|_{l^2([L^2(\Omega)]^d)} + \|\mathbf{u} - \widetilde{\mathbf{u}}_{\Delta t}\|_{l^2([L^2(\Omega)]^d)} \leq C\Delta t^2$$

$$\|\mathbf{u} - \mathbf{u}_{\Delta t}\|_{l^2([H^1(\Omega)]^d)} + \|\mathbf{u} - \widetilde{\mathbf{u}}_{\Delta t}\|_{l^2([H^1(\Omega)]^d)} \leq C\Delta t^{\frac{3}{2}}$$

$$\|p - p_{\Delta t}\|_{l^2(L^2(\Omega))} \leq C\Delta t^{\frac{3}{2}}$$

Proof.

Similar to the proof by Guermond and Shen in [GS04] □

Remark

- $\nabla p^{k+1} \cdot \mathbf{n}|_{\partial\Omega} = (\mathbf{f}(t^{k+1}) - \nu\nabla \times \nabla \times \mathbf{u}^{k+1}) \cdot \mathbf{n}|_{\partial\Omega}$ is a consistent pressure boundary condition
- the splitting-error is of size $\mathcal{O}(\Delta t^2)$

Overview

Weak Formulation

Find $\mathcal{U} = (\mathbf{u}, p) : (0, T) \to \mathbf{V} \times Q = [H^1_0(\Omega)]^d \times L^2_0(\Omega)$, such that

$$(\partial_t \mathbf{u}, \mathbf{v}) + A_G(\mathbf{u}, \mathcal{U}, \mathcal{V}) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathcal{V} = (\mathbf{v}, q) \in \mathbf{V} \times Q$$

where

$$A_G(\mathbf{w}; \mathcal{U}, \mathcal{V}) := a_G(\mathcal{U}, \mathcal{V}) + c(\mathbf{w}; \mathbf{u}, \mathbf{v})$$

$$a_G(\mathcal{U}, \mathcal{V}) := \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + (2\omega \times \mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) + (q, \nabla \cdot \mathbf{u})$$

$$c(\mathbf{w}, \mathbf{u}, \mathbf{v}) := \frac{((\mathbf{w} \cdot \nabla)\mathbf{u}, v) - ((\mathbf{w} \cdot \nabla)\mathbf{v}, \mathbf{u})}{2}$$

Local Projection Stabilization

Idea

- Separate discrete function spaces into small and large scales
- Add stabilization terms only on small scales.

Notations and prerequisites

- Family of shape-regular macro decompositions $\{\mathcal{M}_h\}$
- Let $D_M \subset [L^\infty(M)]^d$ denote a FE space on $M \in \mathcal{M}_h$ for $\mathbf{u}_h$.
- For each $M \in \mathcal{M}_h$, let $\pi_M : [L^2(M)]^d \to D_M$ be the orthogonal L^2-projection.
- $\kappa_M = Id - \pi_h^u$ fluctuation operator
- Averaged streamline direction $\mathbf{u}_M \in \mathbb{R}^d$:
  $|\mathbf{u}_M| \leq C \|\mathbf{u}\|_{L^\infty(M)}, \ \|\mathbf{u} - \mathbf{u}_M\|_{L^\infty(M)} \leq C h_M |\mathbf{u}|_{W^{1,\infty}(M)}$

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Assumptions

Assumption (A.1)

Consider FE spaces (V_h, Q_h) satisfying a discrete inf-sup-condition:

$$\inf_{q\in Q_h\setminus\{0\}} \sup_{v\in V_h\setminus\{0\}} \frac{(\nabla\cdot v, q)}{\|\nabla v\|_{L^2(\Omega)}\|q\|_{L^2(\Omega)}} \geq \beta > 0$$

$$\Rightarrow \quad \mathbf{V_h}^{div} := \{v_h \in V_h \mid (\nabla\cdot v_h, q_h) = 0 \ \ \forall q_h \in Q_h\} \neq \{0\}$$

Assumption (A.2)

The fluctuation operator $\kappa_M = id - \pi_M$ provides the approximation property (depending on D_M and $s \in \{0, \cdots, k\}$):

$$\|\kappa_M \mathbf{w}\|_{L^2(M)} \leq C h_M^l \|\mathbf{w}\|_{W^{l,2}(M)}, \ \forall w \in W^{l,2}(M), \ M \in \mathcal{M}_h, \ l \leq s.$$

A sufficient condition for (A.2) is $\mathbb{P}_{s-1} \subset D_M$.

Assumptions

Assumption (A.3)

Let the FE space V_h satisfy the local inverse inequality

$$\|\nabla v_h\|_{L^2(M)} \leq C h_M^{-1} \|v_h\|_{L^2(M)} \quad \forall v_h \in V_h,\ M \in \mathcal{M}_h.$$

Assumption (A.4)

There are (quasi-)interpolation operators $j_u \colon V \to V_h$ and $j_p \colon Q \to Q_h$ such that for all $M \in \mathcal{M}_h$, for all $\mathbf{w} \in V \cap [W^{l,2}(\Omega)]^d$ with $2 \leq l \leq k+1$:

$$\|\mathbf{w} - j_u\mathbf{w}\|_{L^2(M)} + h_M \|\nabla(\mathbf{w} - j_u\mathbf{w})\|_{L^2(M)} \leq C h_M^l \|\mathbf{w}\|_{W^{l,2}(\omega_M)}$$

and for all $q \in Q \cap H^l(M)$ with $2 \leq l \leq k$ on a suitable patch $\omega_M \supset M$:

$$\|q - j_p q\|_{L^2(M)} + h_M \|\nabla(q - j_p q)\|_{L^2(M)} \leq C h_M^l \|q\|_{W^{l,2}(\omega_M)}$$
$$\|\mathbf{v} - j_u\mathbf{v}\|_{L^\infty(M)} \leq C h_M |\mathbf{v}|_{W^{1,\infty}(M)} \quad \forall \mathbf{v} \in [W^{1,\infty}(M)]^d.$$

Stabilization Terms

- LPS Streamline upwind Petrov-Galerkin (SUPG)

$$s_h(\mathbf{w}_h;\mathbf{u},\mathbf{v}) := \sum_{M\in\mathcal{M}_h} \tau_M(\mathbf{w}_M)(\kappa_M((\mathbf{w}_M\cdot\nabla)\mathbf{u}),\kappa_M((\mathbf{w}_M\cdot\nabla)\mathbf{v}))_M$$

- div-div

$$t_h(\mathbf{w}_h;\mathbf{u},\mathbf{v}) := \sum_{M\in\mathcal{M}_h} \gamma_M(\mathbf{w}_M)(\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{v})_M$$

- LPS Coriolis stabilization

$$a_h(\mathbf{w}_h;\mathbf{u}_h,\mathbf{v}_h) := \sum_{M\in\mathcal{M}_h} \alpha_M(\mathbf{w}_M)(\kappa(\boldsymbol{\omega}_M\times\mathbf{u_h}),\kappa(\boldsymbol{\omega}_M\times\mathbf{v_h}))_M$$

Stability and Existence

Stabilized Problem

Find $\mathcal{U}_h = (\mathbf{u}_h, p_h) : (0, T) \to \mathbf{V_h}^{div} \times Q_h$, s.t. $\forall \mathcal{V}_h = (\mathbf{v}_h, q_h) \in \mathbf{V_h}^{div} \times Q_h$

$$(\partial_t \mathbf{u}_h, \mathbf{v}_h) + A_G(\mathbf{u}_h; \mathcal{U}_h, \mathcal{V}_h) + (s_h + t_h + a_h)(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)$$

Stability

Define for $\mathcal{V} \in \mathbf{V} \times Q$ the norm $|||\mathcal{V}|||_{LPS}^2 := \nu \|\nabla \mathbf{v}\|^2 + (s_h + t_h + a_h)(\mathcal{V}, \mathcal{V})$.
Then the following stability result holds:

$$\|\mathbf{u}_h(t)\|_{L^2(\Omega)}^2 + \int_0^t |||\mathcal{U}_h(s)|||_{LPS}^2 \, ds \leq \|\mathbf{u}_h(0)\|_{L^2(\Omega)}^2 + 3\|\mathbf{f}\|_{L^2(0,T;L^2(\Omega))}^2$$

Corollary (using the generalized Peano theorem)

$\exists$ discrete solution $\mathbf{u}_h : [0, T] \to \mathbf{V_h}^{div}$ for the LPS model.

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Theorem (A., Lube 2014)

Assume a solution according to

$$\mathbf{u} \in [L^\infty(0,T;W^{1,\infty}(\Omega)) \cap L^2(0,T;[W^{k+1,2}(\Omega))]^d,$$
$$\partial_t \mathbf{u} \in [L^2(0,T;W^{k,2}(\Omega))]^d,\ p \in L^2(0,T;W^{k,2}(\Omega)).$$

Let be $\mathbf{u}_h(0) = j_u \mathbf{u}_0$. Then we obtain for $\mathbf{e}_h = \mathbf{u_h} - j_u \mathbf{u}$:

$$\begin{aligned}
&\|\mathbf{e}_h\|^2_{L^\infty(0,t);L^2(\Omega))} + \int_0^t |||\mathbf{e}_h(\tau)|||^2_{LPS}\, d\tau \\
&\leq C \sum_M h_M^{2k} \int_0^t e^{C_G(\mathbf{u})(t-\tau)} \Bigg[\min\left(\frac{d}{\nu}, \frac{1}{\gamma_M}\right) |p(\tau)|^2_{W^{k,2}(\omega_M)} \\
&\quad + (1 + \nu Re_M^2 + \tau_M |\mathbf{u}_M|^2 + d\gamma_M + \alpha_M \|\boldsymbol{\omega}\|^2_{L^\infty(M)} h_M^2) |\mathbf{u}(\tau)|^2_{W^{k+1,2}(\omega_M)} \\
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\end{aligned}$$

with $Re_M := \frac{h_M \|\mathbf{u}\|_{L^\infty(M)}}{\nu}$, $s \in \{0, \cdots, k\}$ and the Gronwall constant $C_G(\mathbf{u}) = 1 + C|\mathbf{u}|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + Ch\|\mathbf{u}\|^2_{L^\infty(0,T;W^{1,\infty}(\Omega))}$

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Choice of Parameters and Projection Spaces

We achieve a method of order k provided

$$\nu Re_M^2 \leq C \Rightarrow h_M \leq C \frac{\sqrt{\nu}}{\|\mathbf{u}\|_{L^\infty(M)}}$$

$$\tau_M |\mathbf{u}_M|^2 h_M^{2(s-k)} \leq C \Rightarrow \tau_M \leq \tau_0 \frac{h_M^{2(k-s)}}{|\mathbf{u}_M|^2}$$

$$\max\{\frac{1}{\gamma_M}, \gamma_M\} \leq C \Rightarrow \gamma_M = \gamma_0$$

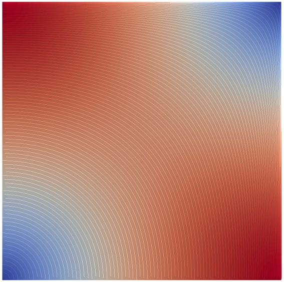
$$\alpha_M \|\boldsymbol{\omega}\|_{L^\infty(M)}^2 h_M^{2(1+s-k)} \leq C \Rightarrow \alpha_M \leq \alpha_0 \frac{h_M^{2(k-s-1)}}{\|\boldsymbol{\omega}\|_{L^\infty(M)}^2}$$

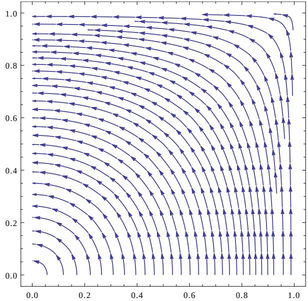
Examples for suitable projection spaces

- One-Level: $\mathbb{Q}_k/\mathbb{Q}_{k-1}/\mathbb{Q}_t$, $\mathbb{P}_k/\mathbb{P}_{k-1}/\mathbb{P}_t$, $\mathbb{Q}_k/\mathbb{P}_{-(k-1)}/\mathbb{P}_t$ $\quad \forall t \leq k-1$
- Two-Level: $\mathbb{P}_k/\mathbb{P}_{k-1}/\mathbb{P}_t$, $\mathbb{Q}_k/\mathbb{Q}_{k-1}/\mathbb{Q}_t$, $\mathbb{Q}_k/\mathbb{P}_{-(k-1)}/\mathbb{P}_t$ $\quad \forall t \leq k-1$

Overview

Couzy Testcase



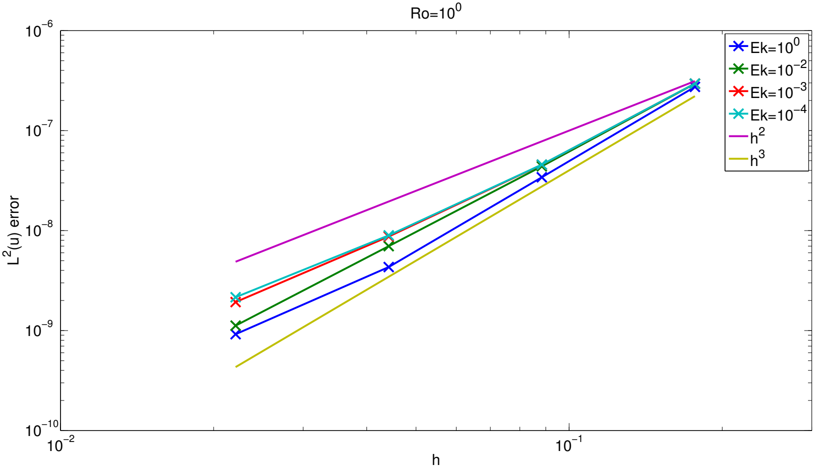


Choose **f** such that the following pair is a solution in $\Omega = [0,1]^2$:

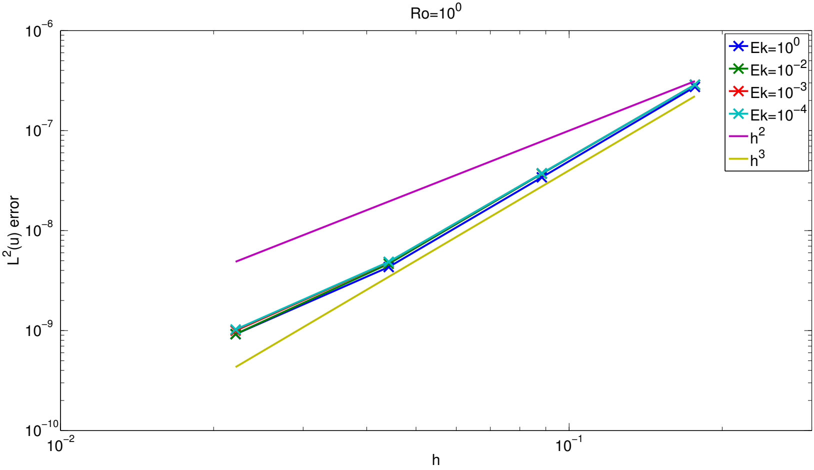
$$\mathbf{u}(x) = \sin(\pi t)\left(-\cos\left(\frac{1}{2}\pi x\right)\sin\left(\frac{1}{2}\pi y\right), \sin\left(\frac{1}{2}\pi x\right)\cos\left(\frac{1}{2}\pi y\right)\right)^T$$

$$p(x) = -\pi\sin\left(\frac{1}{2}\pi x\right)\sin\left(\frac{1}{2}\pi y\right)\sin(\pi t).$$

Couzy Testcase, unstabilized



Couzy Testcase, stabilized



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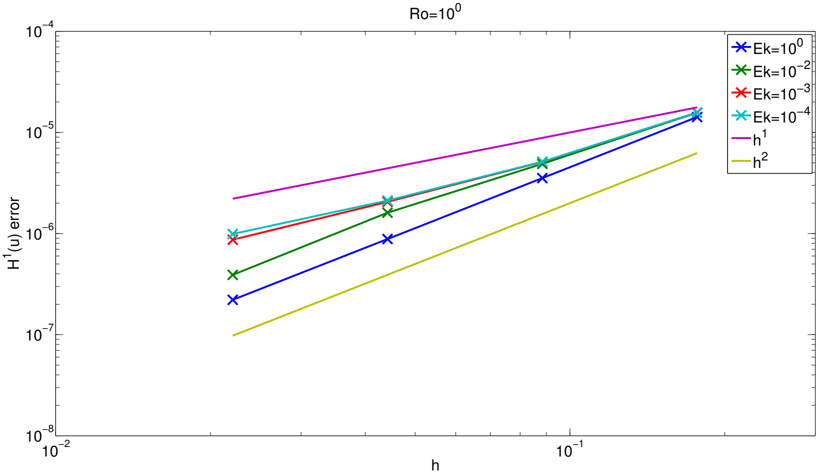
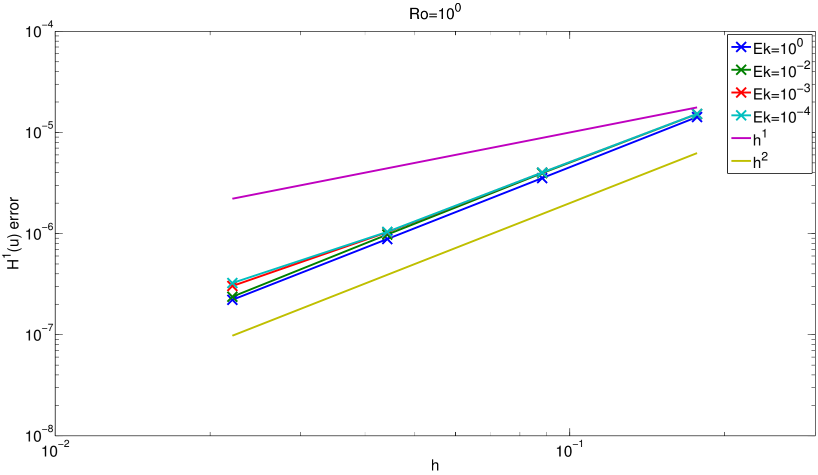
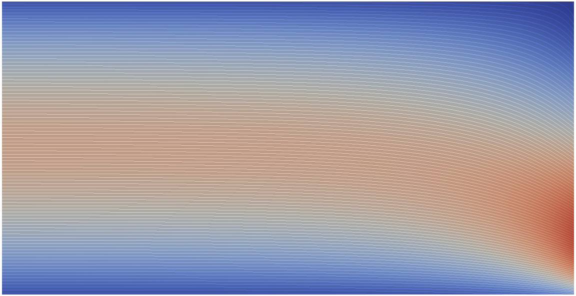


Couzy Testcase, stabilized



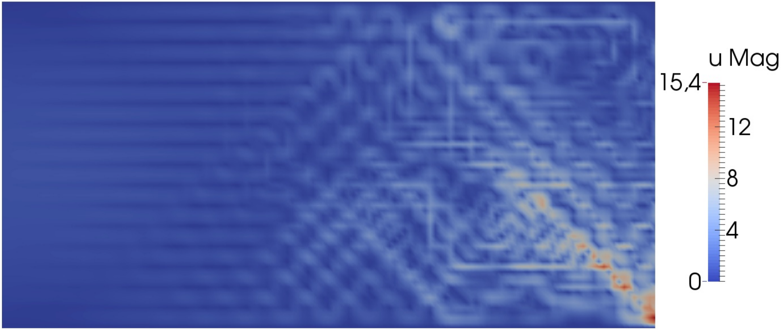
Numerical Results, Rotating Poiseuille Flow

- $\Omega = [-2, 2] \times [-1, 1]$
- $\mathbf{u}(x, y) = \begin{cases} (1 - y^2, 0)^T, & x = -2 \\ (0, 0)^T, & |y| = 1 \end{cases}, \quad (\nabla \mathbf{u} \cdot \mathbf{n})(x = 2, y) = 0$
- $\mathbf{u}_0 = 0, \quad p_0 = 0, \quad \mathbf{f} = 0 \quad \boldsymbol{\omega} = (0, 0, 100), \quad \nu = 10^{-3}$

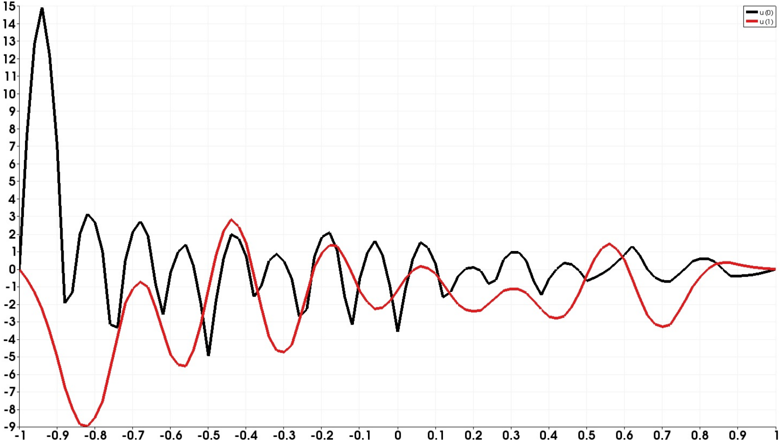


Flow for the parameters $\boldsymbol{\omega} = (0, 0, 1)$, $\nu = 10^{-1}$

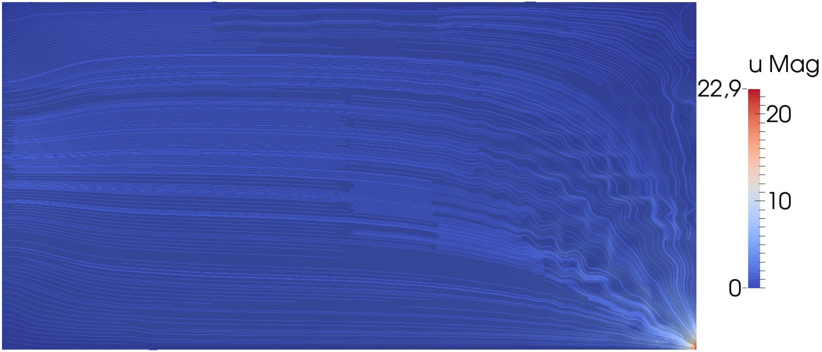
Rotating Poiseuille Flow, div-div



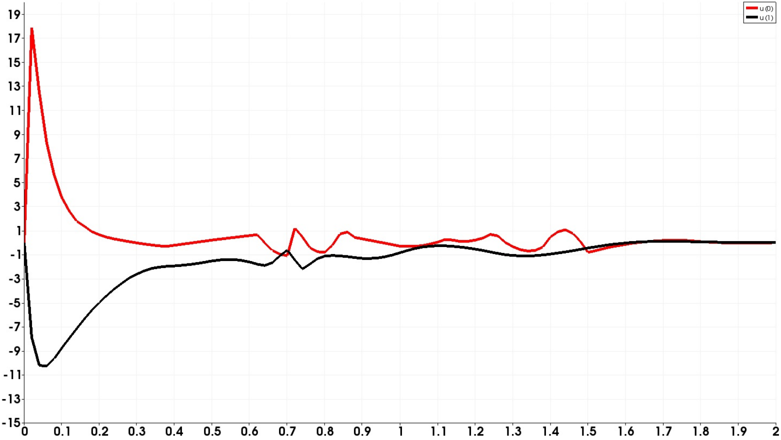
Rotating Poiseuille Flow, div-div



Rotating Poiseuille Flow, div-div adaptive

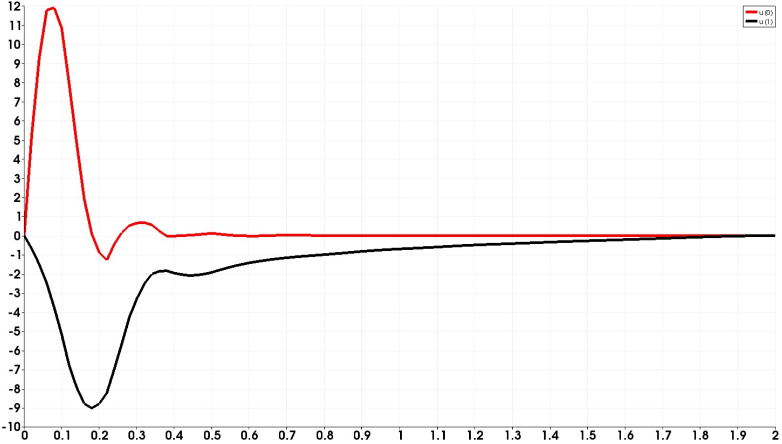


Rotating Poiseuille Flow, div-div adaptive

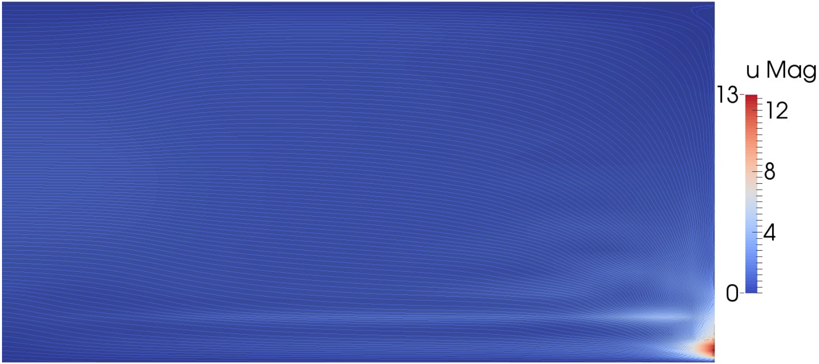


Rotating Poiseuille Flow, Coriolis

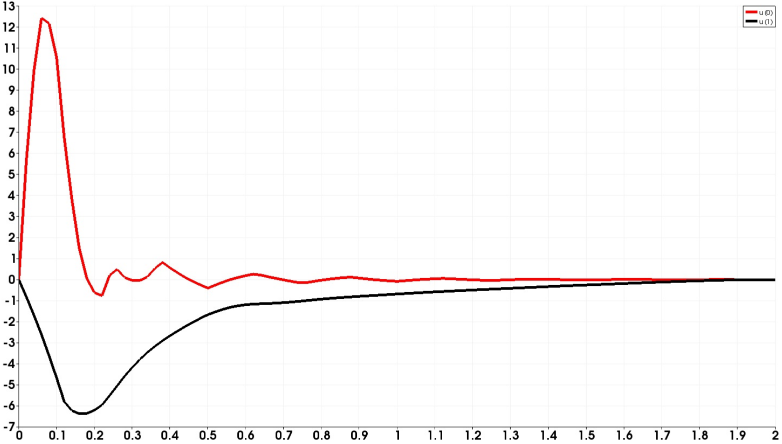
Rotating Poiseuille Flow, Coriolis



Rotating Poiseuille Flow, SUPG

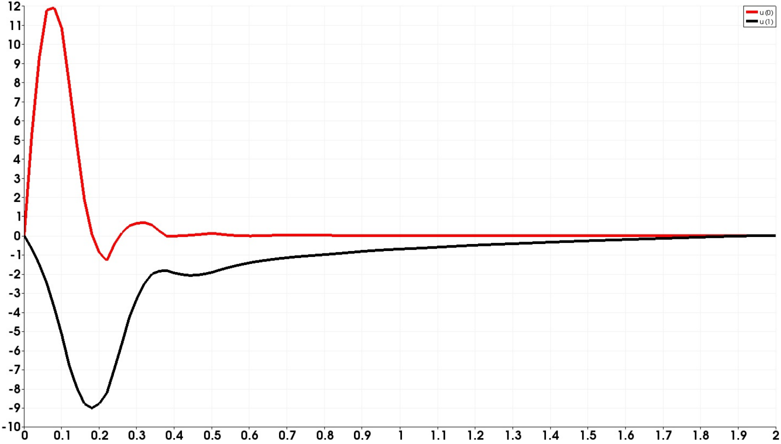


Rotating Poiseuille Flow, SUPG

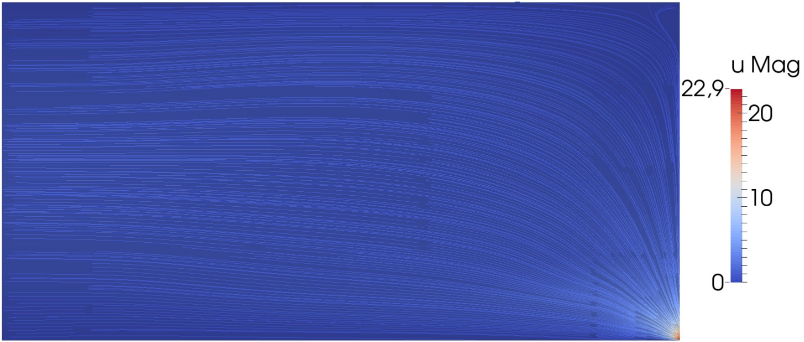


Rotating Poiseuille Flow, SUPG Coriolis

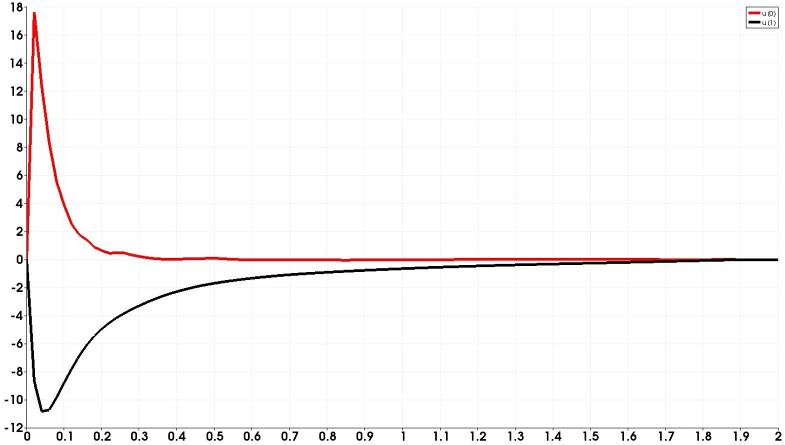
Rotating Poiseuille Flow, SUPG Coriolis



Rotating Poiseuille Flow, SUPG Coriolis Adaptive



Rotating Poiseuille Flow, SUPG Coriolis Adaptive



Summary

Temporal Discretization

Using a BDF2 approach

- Second order convergence in the velocity w.r.t $\|\cdot\|_{l^2([L^2(\Omega)]^d)}$
- Convergence of order $3/2$ in the velocity w.r.t $\|\cdot\|_{l^2([H^1(\Omega)]^d)}$ and in the pressure w.r.t $\|\cdot\|_{l^2([L^2(\Omega)]^d)}$

Spatial Discretization

The Local Projection Stabilization approach provides

- Stability and Existence
- Quasi-optimal error estimates for standard discretizations (e.g. Taylor-Hood)

Numerical results confirm analytical estimates

References

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Local Projection FEM Stabilization for the Time-dependent Incompressible Navier-Stokes Problem.
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Guermond, J ; Shen, Jie:
On the error estimates for the rotational pressure-correction projection methods.
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Thank you for your attention!