Stabilized Finite Element Methods for Rotating Oberbeck-Boussinesq Flow

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Rotating Oberbeck-Boussinesq Model

Momentum Equation

$$\begin{aligned} \partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p + 2\boldsymbol{\omega}\times\boldsymbol{u} &= \boldsymbol{f}_u - \beta\theta\boldsymbol{g}, \\ \nabla\cdot\boldsymbol{u} &= 0, \\ \boldsymbol{u}|_{\partial\Omega} &= \boldsymbol{u}_D, \\ \boldsymbol{u}(t_0) &= \boldsymbol{u}_0 \end{aligned}$$

Heat Equation

$$\begin{aligned} \partial_t\theta - \alpha\Delta\theta + (\boldsymbol{u}\cdot\nabla)\theta &= f_\theta, \\ \theta|_{\partial\Omega} &= \theta_D, \\ \theta(t_0) &= \theta_0 \end{aligned}$$

Rotating Oberbeck-Boussinesq Model

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Heat Equation

$$\begin{aligned} \partial_t \theta - \alpha \Delta \theta + (\boldsymbol{u} \cdot \nabla)\theta &= f_\theta, \\ \theta|_{\partial\Omega} &= \theta_D, \\ \theta(t_0) &= \theta_0 \end{aligned}$$

Weak Formulation

Momentum Equation

Find $(\boldsymbol{u}, p)\colon (t_0, T) \to \boldsymbol{V} \times Q$, such that

$$\begin{aligned}(\partial_t \boldsymbol{u}, \boldsymbol{v}) + c_u(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + (2\omega \times \boldsymbol{u}, \boldsymbol{v})&\\ +\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (p, \nabla \cdot \boldsymbol{v}) + (\beta \boldsymbol{g}\theta, \boldsymbol{v}) &= (\boldsymbol{f}_u, \boldsymbol{v}),\\ (\nabla \cdot \boldsymbol{u}, q) &= 0\end{aligned}$$

holds for all $(\boldsymbol{v}, q) \in \boldsymbol{V} \times Q$ and $t \in (t_0, T)$ a.e.

$$\boldsymbol{V} := [W_0^{1,2}(\Omega)]^d \qquad Q := L_*^2(\Omega)$$

$$c_u(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) := \frac{1}{2}\left[((\boldsymbol{w} \cdot \nabla)\boldsymbol{u}, \boldsymbol{v}) - ((\boldsymbol{w} \cdot \nabla)\boldsymbol{v}, \boldsymbol{u})\right]$$

Weak Formulation

Heat Equation

Find $\theta \in \Theta := W_0^{1,2}(\Omega)$, such that

$$(\partial_t \theta, \psi) + \alpha(\nabla\theta, \nabla\psi) + c_\theta(\boldsymbol{u}; \theta, \psi) = (f_\theta, \psi)$$

holds for all $\psi \in \Theta$ and $t \in (t_0, T)$ a.e.

$$c_\theta(\boldsymbol{w}; \theta, \psi) := \frac{1}{2}\left[((\boldsymbol{w} \cdot \nabla)\theta, \psi) - ((\boldsymbol{w} \cdot \nabla)\psi, \theta)\right]$$

Aim

$$\mathcal{U} := (\boldsymbol{u}, p, \theta), \mathcal{U}_h := (\boldsymbol{u}_h, p_h, \theta_h)$$

- suitable stabilization and choice of parameters
- quasi-optimal error estimates

$$\|\mathcal{U} - \mathcal{U}_h\|^2_{l^\infty(t_0,T;L^2(\Omega))} + |||\mathcal{U} - \mathcal{U}_h|||^2_{l^2(t_0,T;Stab)}$$
$$\leq Ce^{C_G(T-t_0)} \left(\inf_{\mathcal{W}_h} \left\{ \|\mathcal{U} - \mathcal{W}_h\|^2_{l^\infty(t_0,T;L^2(\Omega))} + |||\mathcal{U} - \mathcal{W}_h|||^2_{l^2(t_0,T;Stab)} \right\} \right)$$

- semi-robustness

$$C(\not\alpha, \not\beta, \not\nu, \not\lambda) \qquad C_G(\not\alpha, \not\beta, \not\nu, \not\lambda)$$

- efficiency of the resulting scheme $\longrightarrow$ projection scheme

Local Projection Stabilization

Idea: Stabilize only small scales

- Family of macro decompositions $\{\mathcal{M}_h\}$
- $D_M \subset [L^\infty(M)]^d$ finite element ansatz space on $M \in \mathcal{M}_h$.
- $\pi_M \colon [L^2(M)]^d \to D_M$ orthogonal L^2-projection
- $\kappa_M = Id - \pi_M$ fluctuation operator
- averaged streamline direction $\boldsymbol{u}_M \in \mathbb{R}^d$:
  $|\boldsymbol{u}_M| \leq C\|\boldsymbol{u}\|_{L^\infty(M)}, \; \|\boldsymbol{u} - \boldsymbol{u}_M\|_{L^\infty(M)} \leq Ch_M|\boldsymbol{u}|_{W^{1,\infty}(M)}$

Stabilization Terms

Velocity

$$(\nabla \cdot \boldsymbol{u}_h, q_h) \longrightarrow \tau_{u,gd,M}(\nabla \cdot \boldsymbol{u}_h, \nabla \cdot \boldsymbol{v}_h)_M$$
$$c_u(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) \longrightarrow \tau_{u,SU,M}\big(\kappa((\boldsymbol{u}_M \cdot \nabla)\boldsymbol{u}_h), \kappa(\boldsymbol{u}_M \cdot \nabla)\boldsymbol{v}_h\big)_M$$
$$2(\boldsymbol{\omega} \times \boldsymbol{u}_h, \boldsymbol{v}_h) \longrightarrow \tau_{u,Cor,M}\big(\kappa(\boldsymbol{\omega} \times \boldsymbol{u}_h), \kappa(\boldsymbol{\omega} \times \boldsymbol{v}_h)\big)_M$$

Temperature

$$c_\theta(\boldsymbol{u}_h; \theta_h, \psi_h) \longrightarrow \tau_{\theta_h,SU,M}\big(\kappa((\boldsymbol{u}_M \cdot \nabla)\theta_h), \kappa((\boldsymbol{u}_M \cdot \nabla)\psi_h)\big)_M$$

Assumptions

Interpolation Operators

The interpolation operators $j_u : \boldsymbol{V} \rightarrow \boldsymbol{V}_h$, $j_p : Q \rightarrow Q_h$ and $j_\theta : \Theta \rightarrow \Theta_h$ fulfill for all $M \in \mathcal{M}_h$, $\boldsymbol{w} \in \boldsymbol{V} \cap [H^{l_u}(\Omega)]^d$,$q \in Q \cap H^{l_p}(M)$ and $\psi \in \Theta \cap H^{l_\theta}(M)$

$$\|\boldsymbol{w} - j_u \boldsymbol{w}\|_{L^2(M)} + h_M \|\nabla(\boldsymbol{w} - j_u \boldsymbol{w})\|_{L^2(M)} \leq C h_M^{l_u} \|\boldsymbol{w}\|_{W^{l_u,2}(\omega_M)}$$
$$\|q - j_p q\|_{L^2(M)} + h_M \|\nabla(q - j_p q)\|_{L^2(M)} \leq C h_M^{l_p} \|q\|_{W^{l_p,2}(\omega_M)}$$
$$\|\psi - j_\theta \psi\|_{L^2(M)} + h_M \|\nabla(\psi - j_\theta \psi)\|_{L^2(M)} \leq C h_M^{l_\theta} \|\psi\|_{W^{l_\theta,2}(\omega_M)}$$

on a patch $\omega_M \supset M$. Furthermore it holds $j_u(\boldsymbol{V}^{div}) \subset \boldsymbol{V}_h^{div}$ and

$$\|\boldsymbol{v} - j_u \boldsymbol{v}\|_{L^\infty(M)} \leq C h_M |\boldsymbol{v}|_{W^{1,\infty}(\omega_M)} \quad \forall \boldsymbol{v} \in [W^{1,\infty}(\omega_M)]^d.$$

Assumptions

Fluctuation Operators

For all $M \in \mathcal{M}_h$ it holds

- κ_M^u: $\boldsymbol{V} \to \boldsymbol{V}$ fulfills for all $\boldsymbol{w} \in \boldsymbol{V} \cap [H^{l_u}(\Omega)]^d$ and $l_u \leq s_u \leq k_u$.
$$\|\kappa_M \boldsymbol{w}\|_{L^2(M)} \leq C h_M^{l_u} \|\boldsymbol{w}\|_{W^{l_u,2}(M)}$$
- κ_M^θ: $\Theta \to \Theta$ fulfills for all $\psi \in \Theta \cap H^{l_\theta}(M)$ and $l_\theta \leq s_\theta \leq k_\theta$
$$\|\kappa_M \psi\|_{L^2(M)} \leq C h_M^{l_\theta} \|\psi\|_{W^{l_\theta,2}(M)}.$$

Inf-Sup Stability

Consider inf-sup stable ansatz spaces $(\boldsymbol{V}_h, Q_h)$:

$$\inf_{q \in Q_h \setminus \{0\}} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h \setminus \{\boldsymbol{0}\}} \frac{(\nabla \cdot \boldsymbol{v}_h, q_h)}{\|\nabla \boldsymbol{v}_h\|_{L^2(\Omega)} \|q_h\|_{L^2(\Omega)}} \geq \beta_{u,h}(\not{h}) > 0$$

$$\Rightarrow \quad \boldsymbol{V}_h^{div} := \{\boldsymbol{v}_h \in \boldsymbol{V}_h \mid (\nabla \cdot \boldsymbol{v}_h, q_h) = 0 \ \forall q_h \in Q_h\} \neq \{\boldsymbol{0}\}$$

Discretization in Time - Velocity

Find $\widetilde{\boldsymbol{u}}_{ht}^n \in \boldsymbol{V}_h$, such that

$$
\begin{aligned}
&(D_t(\widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{u}_{ht}^n), \boldsymbol{v}_h) + (2\boldsymbol{\omega}^n \times \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) + \nu(\nabla \widetilde{\boldsymbol{u}}_{ht}^n, \nabla \boldsymbol{v}_h) + c_u(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) \\
&\quad + s_{u,SU}(\widetilde{\boldsymbol{u}}_{ht}^n; \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) + s_{u,Cor}(\boldsymbol{\omega}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) + s_{u,gd}(\widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) \\
&= (\boldsymbol{f}_u^n, \boldsymbol{v}_h) - (\nabla p_{ht}^{n-1}, \boldsymbol{v}_h) - \beta(\boldsymbol{g}^n \theta_{ht}^{*,n}, \boldsymbol{v}_h)
\end{aligned}
$$

holds for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$.

$$
D_t(\widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{u}_{ht}^n) := \frac{3\widetilde{\boldsymbol{u}}_{ht}^n - 4\boldsymbol{u}_{ht}^{n-1} + \boldsymbol{u}_{ht}^{n-2}}{2\Delta t} \qquad f^{*,n} := 2f^{n-1} - f^{n-2}
$$

Discretization in Time - Pressure and Temperature

Find $(\boldsymbol{u}_{ht}^n, p_{ht}^n) \in \boldsymbol{V}_h^{div} \times Q_h$, such that

$$\left(\frac{3\boldsymbol{u}_{ht}^n - 3\widetilde{\boldsymbol{u}}_{ht}^n}{2\Delta t} + \nabla(p_{ht}^n - p_{ht}^{n-1}), \boldsymbol{y}_h\right) = 0,$$
$$(\nabla \cdot \boldsymbol{u}_{ht}^n, q_h) = 0$$

holds for all $\boldsymbol{y}_h \in \nabla Q_h \oplus \boldsymbol{V}_h^{div}$, $q_h \in Q_h$.

Find $\theta_{ht}^n \in \Theta_h$, such that

$$D_t(\theta_{ht}^n, \psi_h) + \alpha\left(\nabla\theta_{ht}^n, \nabla\psi_h\right) + c_\theta(\widetilde{\boldsymbol{u}}_{ht}^n; \theta_{ht}^n, \psi_h)$$
$$+ s_{\theta,SU}(\widetilde{\boldsymbol{u}}_{ht}^n; \theta_{ht}^n, \psi_h) = (f_\theta^n, \psi_h)$$

holds for all $\psi_h \in \Theta_h$.

Strategy

- Split error according to

$$\boldsymbol{\eta}_u^n = \boldsymbol{u}(t_n) - j_u\boldsymbol{u}(t_n) \quad \boldsymbol{e}_u^n = j_u\boldsymbol{u}(t_n) - \widetilde{\boldsymbol{u}}_{ht}^n \quad \boldsymbol{\xi}_u^n = \boldsymbol{\eta}_u^n + \boldsymbol{e}_u^n$$

- Estimate the discretization error in each equation separately
- Combine discretization error estimates
- Combine with with interpolation errors $\Rightarrow$ total error
- Assumption on interpolation operators $\boldsymbol{\eta} \Rightarrow$ convergence

$$|||\boldsymbol{u}|||_{LPS_u}^2 := \nu\|\nabla \boldsymbol{u}\|_0^2 + s_{u,gd}(\boldsymbol{u}, \boldsymbol{u}) + s_{u,SU}(\widetilde{\boldsymbol{u}}_{ht}; \boldsymbol{u}, \boldsymbol{u}) + s_{u,Cor}(\boldsymbol{\omega}; \boldsymbol{u}, \boldsymbol{u})$$
$$|||\theta|||_{LPS_\theta}^2 := \alpha\|\nabla \theta\|_0^2 + s_{\theta,SU}(\widetilde{\boldsymbol{u}}_{ht}, \theta)$$

Convective Terms I

Lemma

Assuming $\boldsymbol{u} \in L^\infty(t_0, T; [W^{1,\infty}(\Omega)]^d)$ *the difference of the convective terms can be bounded by*

$$
\begin{aligned}
&c(\boldsymbol{u}(t_n); \boldsymbol{u}(t_n), \boldsymbol{e}_u^n) - c(\widetilde{\boldsymbol{u}}_{ht}^n; \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{e}_u^n) \\
&\leq \frac{1}{4\epsilon} \sum_M \frac{1 + \nu Re_M^2}{h_M^2} \|\boldsymbol{\eta}_u^n\|_{L^2(M)}^2 + 3\epsilon |||\boldsymbol{\eta}_u^n|||_{LPS}^2 + 4\epsilon |||\boldsymbol{e}_u^n|||_{LPS}^2 \\
&\quad + \left[|\boldsymbol{u}(t_n)|_{W^{1,\infty}(\Omega)} + \left(\epsilon h^2 + \frac{C}{\epsilon} \max_M \frac{h^2}{\gamma_M} \right) \|\boldsymbol{u}(t_n)\|_{W^{1,\infty}(\Omega)}^2 \right] \|\boldsymbol{e}_u^n\|_{L^2(\Omega)}^2
\end{aligned}
$$

with the local Reynolds number $Re_M := h_M \|\boldsymbol{u}(t_n)\|_{L^\infty(M)} / \nu$.

Convective Terms II

Lemma

Assuming $\boldsymbol{u} \in L^\infty(t_0, T; [W^{1,\infty}(\Omega)]^d)$ *and* $\widetilde{\boldsymbol{u}}_{ht} \in l^\infty(t_0, T; [L^\infty(\Omega)]^d)$ *the difference of the convective terms can be bounded by*

$$
\begin{aligned}
&c(\boldsymbol{u}(t_n); \boldsymbol{u}(t_n), \boldsymbol{e}_u^n) - c(\widetilde{\boldsymbol{u}}_{ht}^n; \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{e}_u^n) \\
&\leq \Bigg[|\boldsymbol{u}(t_n)|_{W^{1,\infty}} + \epsilon h^{2r} |\boldsymbol{u}(t_n)|_{W^{1,\infty}}^2 + \frac{C}{\epsilon} \frac{h^{2r}}{\gamma} |\boldsymbol{u}(t_n)|_{W^{1,\infty}}^2 \\
&\qquad + \frac{C}{\epsilon} \frac{h^{2r-2}}{\gamma} \|\boldsymbol{u}(t_n)\|_\infty^2 + \epsilon h^{2r-2} \|\widetilde{\boldsymbol{u}}_{ht}^n\|_\infty^2 \Bigg] \|\boldsymbol{e}_u^n\|_0^2 \\
&\quad + \frac{C}{\epsilon} h^{-2r} \|\boldsymbol{\eta}_u^n\|_0^2 + 3\epsilon h^{2-2r} |||\boldsymbol{\eta}_u^n|||_{LPS}^2 + 3\epsilon h^{2-2r} |||\boldsymbol{e}_u^n|||_{LPS}^2
\end{aligned}
$$

where $C(\not h, \not\epsilon, \not\nu)$ *and* $r \in \{0, 1\}$.

Convergence

Theorem (Convergence LPS)

$$\|\widetilde{\xi}_u\|^2_{l^\infty(t_0,T;L^2)} + \|\widetilde{\xi}_\theta\|^2_{l^\infty(t_0,T;L^2)} + \|\widetilde{\xi}_u\|^2_{l^2(t_0,T;LPS_u)} + \|\widetilde{\xi}_\theta\|^2_{l^2(t_0,T;LPS_\theta)}$$
$$\lesssim e^{T-t_0}\left(\Delta t^2 + h^{2k_u} + h^{2k_p+2} + h^{2k_\theta}\right)$$

provided

$$\max_{M\in\mathcal{M}_h}\{\tau^n_{u,SU,M}|\widetilde{\boldsymbol{u}}^n_M|^2\} \lesssim \nu h^{2k_u-2s_u} \qquad \tau^n_{u,gd} \sim 1$$
$$\max_{M\in\mathcal{M}_h}\{\tau^n_{\theta,SU,M}|\widetilde{\boldsymbol{u}}^n_M|^2\} \lesssim h^{2k_\theta-2s_\theta} \qquad \tau^n_{u,Cor}|\boldsymbol{\omega}^n|^2 \lesssim h^{2k_u-2-2s_u}$$

Convergence L^2

Theorem (Convergence L^2)

For the L^2 error it holds the improved error estimate

$$\|\widetilde{\boldsymbol{\xi}}_u\|^2_{l^\infty(t_0,T;L^2)} + \|\widetilde{\boldsymbol{\xi}}_\theta\|^2_{l^\infty(t_0,T;L^2)}$$
$$\lesssim e^{T-t_0}\left((\Delta t)^2 + h^{2k_u+2} + h^{2k_p+4} + h^{2k_\theta+2}\right)$$

provided $\Delta t \lesssim h^2$ and

$$\tau^n_{u,gd} \sim 1 \qquad \max_{M\in\mathcal{M}_h}\{\tau^n_{u,SU,M}|\widetilde{\boldsymbol{u}}^n_M|^2\} \lesssim \nu h^{2+2k_u-2s_u}$$
$$\tau^n_{u,Cor}|\boldsymbol{\omega}^n|^2 \lesssim h^{2k_u-2s_u} \qquad \max_{M\in\mathcal{M}_h}\{\tau^n_{\theta,SU,M}|\widetilde{\boldsymbol{u}}^n_M|^2\} \lesssim h^{2+2k_\theta-2s_\theta}$$

Convergence L^2

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For the L^2 error it holds the improved error estimate

$$\|\widetilde{\xi}_u\|^2_{l^\infty(t_0,T;L^2)} + \|\widetilde{\xi}_\theta\|^2_{l^\infty(t_0,T;L^2)} \lesssim e^{\frac{T-t_0}{1-K}} \left(\frac{(\Delta t)^4}{\nu^3} + h^{2k_u+2} + h^{2k_p+4} + h^{2k_\theta+2} \right)$$

provided $\Delta t \lesssim h^2$ and

$$\tau^n_{u,gd} \sim 1 \qquad \max_{M\in\mathcal{M}_h} \{\tau^n_{u,SU,M} |\widetilde{\boldsymbol{u}}^n_M|^2\} \lesssim \nu h^{2+2k_u-2s_u}$$

$$\tau^n_{u,Cor} |\boldsymbol{\omega}^n|^2 \lesssim h^{2k_u-2s_u} \qquad \max_{M\in\mathcal{M}_h} \{\tau^n_{\theta,SU,M} |\widetilde{\boldsymbol{u}}^n_M|^2\} \lesssim h^{2+2k_\theta-2s_\theta}$$

$$K := \frac{\Delta t}{\nu} + \Delta t \frac{h^{4k_u}}{\nu^3} \lesssim 1$$

Convergence - Pressure

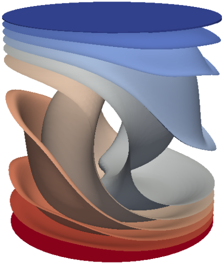
Theorem (Pressure)

The total pressure error can be bounded by

$$\|\xi_p\|^2_{l^2(t_0,T;L^2)} \leq \frac{\|\widetilde{\xi}_u^n\|^2_{l^\infty(t_0,T;L^2)}}{(\Delta t)^2} + \|\widetilde{\xi}_u^n\|^2_{l^2(t_0,T;LPS_u)} + \|\xi_\theta^n\|^2_{l^2(t_0,T;L^2)} + (\Delta t)^4$$

provided the assumptions for the estimates on the LPS errors are fulfilled.

Rayleigh-Bénard Convection



$$Ra = |\boldsymbol{g}|\beta\Delta\theta H^3/(\nu\alpha)$$
$$Pr = \nu/\alpha$$
$$Ro = U_{ref}/(2|\boldsymbol{\omega}|H)$$

Flow driven by temperature gradient

Dirichlet boundary conditions for upper and lower boundary:

$$\theta_{bottom} = 0.5, \quad \theta_{top} = -0.5,$$

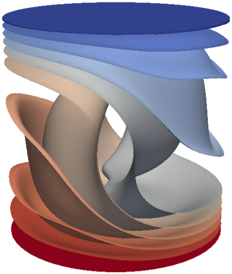
and isolating hull:

$$\boldsymbol{n} \cdot \nabla\theta|_{r=0.5} = 0$$

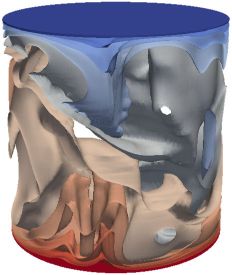
No-slip boundary conditions for the velocity:

$$\boldsymbol{u} = \boldsymbol{0}|_{\partial\Omega}$$

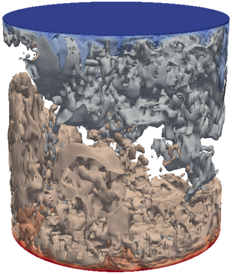
Rayleigh-Bénard Convection - Iso Surfaces and Benchmark



$Ra = 10^5, Pr = 0.786$



$Ra = 10^7, Pr = 0.786$



$Ra = 10^9, Pr = 0.786$

- Nusselt number as measure for $\frac{\text{heat flux}}{\text{heat conduction}}$

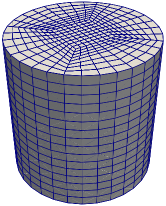
$$Nu(z_0, t) := \frac{L}{\alpha A \Delta\theta} \langle u_z\theta - \alpha\partial_z\theta \rangle_{z=z_0}(z_0, t)$$

- $Nu(z_0, t) = Nu(t)$ for stationary solutions and in the time average

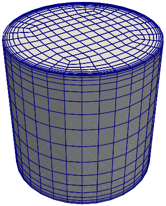
$$\sigma := \max\{|Nu^{\text{avg}} - Nu(z)|, z \in \{-0.5, -0.25, 0, 0.25, 0.5\}\}$$

Parameter Design, $Pr = 0.786$, $Ra = 10^9$ ($N = 10 \cdot 8^3$)

$\tau_{u,gd}$	$\tau_{u,SU}$	$\tau_{\theta,SU}$	$Nu^{avg}_{ld,th}$	$\sigma_{ld,th}$	$Nu^{avg}_{ld,bb}$	$\sigma_{ld,bb}$	Nu^{ref}
0.01	0	0	41.46	40.20	47.53	23.40	63.1
0.01	hu1	0	38.71	43.03	44.30	24.79	
0.01	0	hu1	37.61	10.84	54.26	16.53	
0.01	hu1	hu1	37.05	10.31	49.13	12.92	



$\tau_{u,gd}$	$\tau_{u,SU}$	$\tau_{\theta,SU}$	Nu^{avg}_{ht}	σ_{ht}	Nu^{avg}_{bb}	σ_{bb}	Nu^{ref}
0	0	0	118.79	137.56			63.1
0.01	0	0	55.52	1.35	58.14	1.48	
0.01	hu1	0	53.84	1.41	58.27	1.47	
0.01	0	hu1	52.45	3.48	56.53	3.06	
0.01	hu1	hu1	51.81	3.43	54.04	3.33	

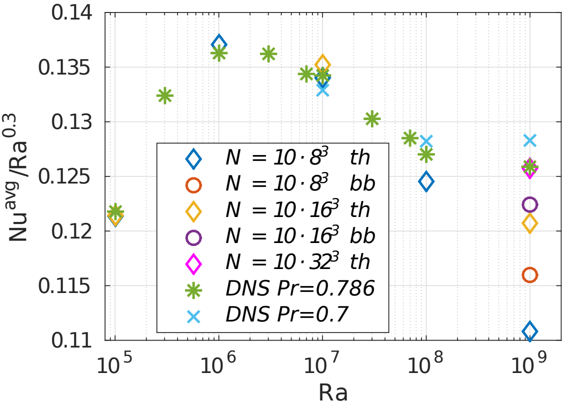


th : Taylor-Hood elements *bb* : bubble-enriched Taylor-Hood elements

hu1: $\tau_{u/\theta,SU} = \frac{h}{2}\|\boldsymbol{u}_h\|_{\infty,M}$

Rayleigh-Bénard - $Pr = 0.786$, Mesh Convergence

$N = 10 \cdot 32^3 \Rightarrow 1.1 \cdot 10^7\, DoFs$ vs. DNS with $1.5 \cdot 10^9\, DoFs$



S. Wagner, O. Shishkina und C. Wagner. „Boundary layers and wind in cylindrical Rayleigh–Bénard cells“. In: *Journal of Fluid Mechanics* 697 (2012), S. 336–366

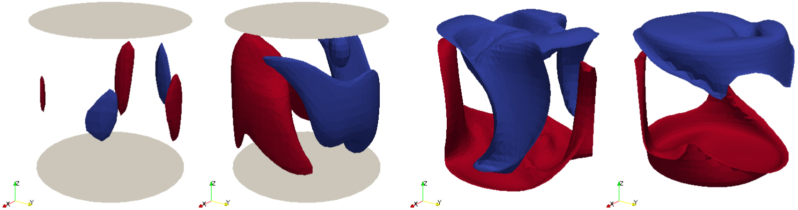
Rotating Rayleigh-Bénard Convection, $Pr = 6.4$, $N = 10 \cdot 8^3$

Ra	Ro	Nu^{avg}	σ	$\tau_{u,gd,M}$	mesh	Nu^{ref}
10^7	0.09	15.7992	0.2971	0.1	aniso	16.1 ± 0.5
10^7	0.36	18.9784	0.1057	0.1	aniso	18.8 ± 0.4
10^7	1.08	17.3130	0.0620	0.1	aniso	17.4 ± 0.3
10^7	∞	16.4804	0.1806	0.1	aniso	16.5 ± 0.2
10^8	0.09	38.8861	0.6861	0.1	aniso	38.2 ± 0.8
10^8	∞	32.0387	0.5651	0.1	aniso	33.2 ± 0.4
10^9	0.09	64.8679	6.5222	0.1	aniso	73.8 ± 1.0
10^9	0.36	78.2142	5.9778	0.01	aniso	72.2 ± 0.9
10^9	1.08	71.8906	2.9568	0.01	aniso	67.0 ± 1.6
10^9	∞	66.2219	2.6035	0.01	aniso	66.5 ± 1.8

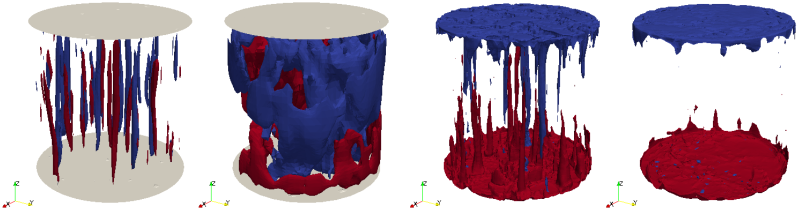
DNS results from:

GL Kooij, MA Botchev und BJ Geurts. „Direct numerical simulation of Nusselt number scaling in rotating Rayleigh–Bénard convection“. In: *International Journal of Heat and Fluid Flow* 55 (2015), S. 26–33

Rayleigh-Bénard Convection - $Pr = 6.4$, $Ra \in \{10^6, 10^9\}$

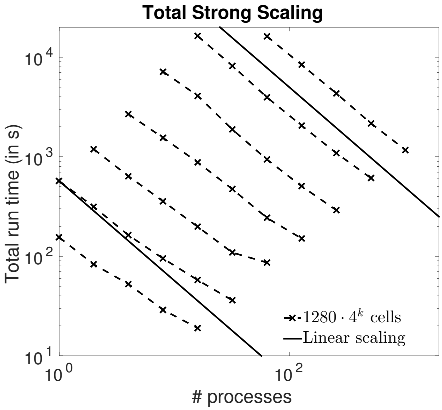


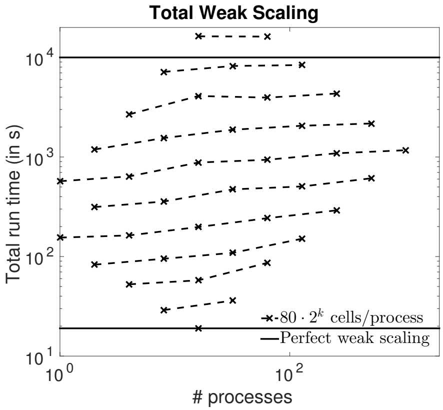
$u_Z = \pm 0.007$, $Ro \in \{0.09, \infty\}$ $\theta = \pm 0.15$, $Ro \in \{0.09, \infty\}$



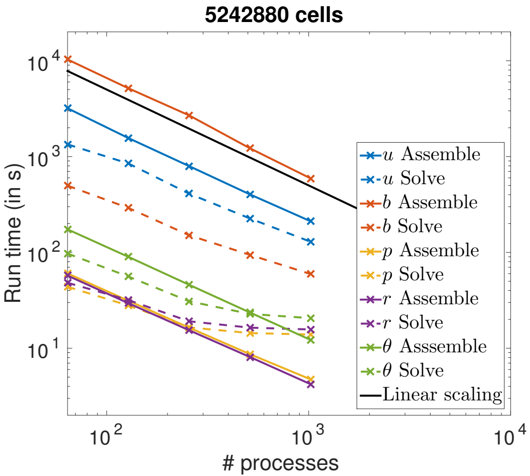
$$Ra = |\boldsymbol{g}|\beta\Delta\theta H^3/(\nu\alpha) \qquad Pr = \nu/\alpha \qquad Ro = U_{ref}/(2|\boldsymbol{\omega}|H)$$

Strong Scaling





Detailed Scaling



Summary

Analysis for the fully discretized model:

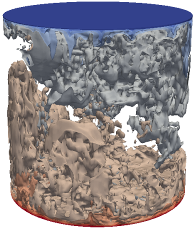
- quasi-optimal error estimates
- semi-robust error estimates
- extension to variable time step size possible

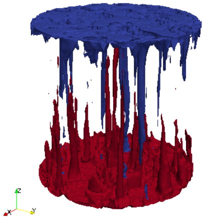
Numerical results:

- grad-div stabilization essential
- LPS stabilizations
  - diminish unphysical oscillations
  - negligible for suitably problem adapted meshes
  - suitable as simple subgrid model
- convincing scaling results for the implementation
- bubble-enrichment always beneficial

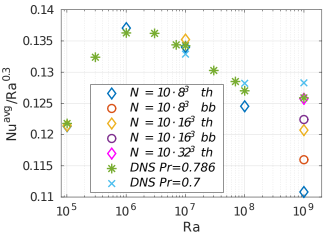
References

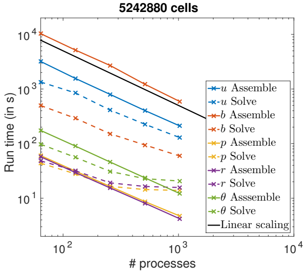
- Daniel Arndt. „Stabilized Finite Element Methods for Coupled Incompressible Flow Problems“. Diss. Georg-August University Göttingen, 2016.
- Daniel Arndt, Helene Dallmann und Gert Lube. „Quasi-Optimal Error Estimates for the Fully Discretized Stabilized Incompressible Navier-Stokes Problem“. In: *ESAIM: Mathematical Modelling and Numerical Analysis* (2015). revised.
- Helene Dallmann und Daniel Arndt. „Stabilized Finite Element Methods for the Oberbeck–Boussinesq Model“. In: *Journal of Scientific Computing* (2016), S. 1–30. ISSN: 1573-7691. DOI: 10.1007/s10915-016-0191-z. URL: http://dx.doi.org/10.1007/s10915-016-0191-z.





Thank you for your attention!





Compatibility Condition

Lemma

Let $\boldsymbol{V}_h(M) = \{\boldsymbol{v}_h|_M : \ \boldsymbol{v}_h \in \boldsymbol{V}_h, \ \boldsymbol{v}_h = 0 \text{ on } \Omega \setminus M\}$ *fulfill the compatibility condition*

$$\exists \beta_u > 0 : \quad \inf_{\boldsymbol{w}_h \in D_M} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h(M)} \frac{(\boldsymbol{v}_h, \boldsymbol{w}_h)_M}{\|\boldsymbol{v}_h\|_{L^2(M)} \|\boldsymbol{w}_h\|_{L^2(M)}} \geq \beta_u(\cancel{h}, \cancel{M}).$$

Then there exists an interpolation operator $i_u : \boldsymbol{V} \to \boldsymbol{V}_h$ *such that it holds for all* $1 \leq l_u \leq k_u + 1$, $\boldsymbol{v} \in V \cap [W^{l_u,2}(\Omega)]^d$ *and* $\boldsymbol{w}_h \in D_M$:

$$(\boldsymbol{v} - i_u \boldsymbol{v}, \boldsymbol{w}_h) = 0$$

$$\|\boldsymbol{v} - i_u \boldsymbol{v}\|_{L^2(M)} + h_M |\boldsymbol{v} - i_u \boldsymbol{v}|_{W^{1,2}(M)} \leq C h_M^{l_u} \|\boldsymbol{v}\|_{W^{l_u,2}(\omega_M)}.$$

G. Matthies, P. Skrzypacz und L. Tobiska. „A unified convergence analysis for local projection stabilisations applied to the Oseen problem“. In: *ESAIM-Mathematical Modelling and Numerical Analysis* 41.4 (2007), S. 713–742

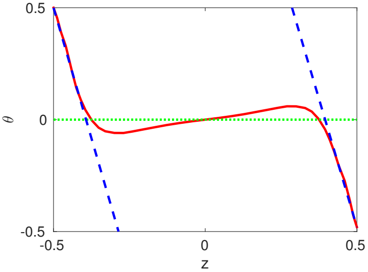
Convective Terms II

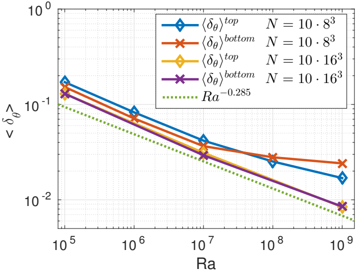
Lemma

Provided the compatibility condition holds true and assuming $\boldsymbol{u}(t_n) \in [W^{1,\infty}(\Omega)]^d$, $\widetilde{\boldsymbol{u}}_{ht}^n \in [W^{1,\infty}(\Omega)]^d$ the difference of the convective terms can be bounded by

$$
\begin{aligned}
&c(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{e}_u^n) - c(\widetilde{\boldsymbol{u}}_{ht}; \widetilde{\boldsymbol{u}}_{ht}, \boldsymbol{e}_u^n) \\
&\leq \frac{1}{2\epsilon} \sum_M \Big(\frac{1}{\tau_M} + \frac{1}{2h_M^2}\Big) \|\boldsymbol{\eta}_u^n\|_{L^2(M)}^2 + 3\epsilon |||\boldsymbol{\eta}_u^n|||_{LPS}^2 + 4\epsilon |||\boldsymbol{e}_u^n|||_{LPS}^2 \\
&\quad + C\Big[|\boldsymbol{u}|_{W^{1,\infty}(\Omega)} + \Big(\Big(\epsilon h^2 + \frac{1}{\epsilon} \max_M \frac{h^2}{\gamma_M}\Big) |\boldsymbol{u}|_{W^{1,\infty}(M)}^2\Big) \\
&\quad + \epsilon \tau_M |\widetilde{\boldsymbol{u}}_{ht}|_{W^{1,\infty}(M)}^2\Big] \|\boldsymbol{e}_h\|_{L^2(\Omega)}^2.
\end{aligned}
$$

Rayleigh-Bénard - Thermal Boundary Layer Thickness





- Anisotropic Mesh using $\mathbb{Q}_2/\mathbb{Q}_1/\mathbb{Q}_2$ - Elementen
- $\gamma_M = 0.1$ für $Ra \in \{10^5, 10^7\}$ und $\gamma_M = 0.01$ for $Ra = 10^9$
- $\langle\delta_\theta\rangle \propto Ra^{-0.285}$

S. Wagner, O. Shishkina und C. Wagner. „Boundary layers and wind in cylindrical Rayleigh–Bénard cells“. In: *Journal of Fluid Mechanics* 697 (2012), S. 336–366

Rayleigh-Bénard - Non-Dimensional Formulation

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \frac{Pr^{1/2}}{Ra^{1/2}\Gamma^{3/2}}\Delta \boldsymbol{u} + \theta \boldsymbol{e}_z + \frac{\Gamma^{1/2}}{Ro}\boldsymbol{e}_z \times \boldsymbol{u} + Fr\theta \boldsymbol{e}_z \times (\boldsymbol{e}_z \times \boldsymbol{r})$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\partial_t \theta + (\boldsymbol{u} \cdot \nabla)\theta = -\frac{\Delta\theta}{Pr^{1/2}Ra^{1/2}\Gamma^{3/2}}$$

$$Ra = |\boldsymbol{g}|\beta\Delta\theta H^3/(\nu\alpha)$$

$$Pr = \nu/\alpha$$

$$Ro = U/(2|\boldsymbol{\omega}|H)$$

$$Fr = |\boldsymbol{\omega}|^2 H/\boldsymbol{g} \ll 1$$

- $L = H$ reference length
- $D = \Gamma H$ cylinder diameter
- $U = \sqrt{|\boldsymbol{g}|\beta\Delta\theta D}$ free fall velocity
- $T = L_{ref}/U$ characteristic time