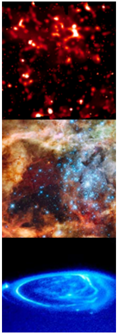
Suitability of LPS for Laminar and Turbulent Flow



Daniel Arndt
Helene Dallmann

Georg-August-Universität Göttingen
Institute for Numerical and Applied Mathematics

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Overview

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Setting

Navier Stokes Equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega \times (0, T]$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega \times (0, T]$$

$\Omega \subset \mathbb{R}^d$ bounded polyhedral domain

Averaged Navier Stokes Equations

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + (\overline{\mathbf{u}} \cdot \nabla)\overline{\mathbf{u}} \nabla \cdot \tau - \nu \Delta \overline{\mathbf{u}} + \nabla p = \mathbf{f} \qquad \text{in } \Omega \times (0, T]$$

$$\nabla \cdot \overline{\mathbf{u}} = 0 \qquad \text{in } \Omega \times (0, T]$$

Reynolds subgrid tensor $\tau = \overline{\mathbf{u}\mathbf{u}^T} - \overline{\mathbf{u}}\overline{\mathbf{u}}^T = \overline{(\mathbf{u} - \overline{\mathbf{u}})(\mathbf{u} - \overline{\mathbf{u}})^T}$

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4. Summary

Weak Formulation

Find $\mathcal{U} = (\mathbf{u}, p) : (0, T) \to \mathbf{V} \times Q = [H_0^1(\Omega)]^d \times L_0^2(\Omega)$, such that

$$(\partial_t \mathbf{u}, \mathbf{v}) + A_G(\mathbf{u}, \mathcal{U}, \mathcal{V}) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathcal{V} = (\mathbf{v}, q) \in \mathbf{V} \times Q$$

where

$$\begin{aligned} A_G(\mathbf{w}; \mathcal{U}, \mathcal{V}) &:= a_G(\mathcal{U}, \mathcal{V}) + c(\mathbf{w}; \mathbf{u}, \mathbf{v}) \\ a_G(\mathcal{U}, \mathcal{V}) &:= \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) + (q, \nabla \cdot \mathbf{u}) \\ c(\mathbf{w}, \mathbf{u}, \mathbf{v}) &:= \frac{((\mathbf{w} \cdot \nabla)\mathbf{u}, v) - ((\mathbf{w} \cdot \nabla)\mathbf{v}, \mathbf{u})}{2} \end{aligned}$$

Local Projection Stabilization

Idea

- Separate discrete function spaces into small and large scales
- Add stabilization terms only on small scales.

Notations and prerequisites

- Family of shape-regular macro decompositions $\{\mathcal{M}_h\}$
- Let $D_M \subset [L^\infty(M)]^d$ denote a FE space on $M \in \mathcal{M}_h$ for $\mathbf{u}_h$.
- For each $M \in \mathcal{M}_h$, let $\pi_M \colon [L^2(M)]^d \to D_M$ be the orthogonal L^2-projection.
- $\kappa_M = Id - \pi_h^u$ fluctuation operator
- Averaged streamline direction $\mathbf{u}_M \in \mathbb{R}^d$:
  $|\mathbf{u}_M| \leq C\|\mathbf{u}\|_{L^\infty(M)}, \ \|\mathbf{u} - \mathbf{u}_M\|_{L^\infty(M)} \leq Ch_M|\mathbf{u}|_{W^{1,\infty}(M)}$

Assumptions

Assumption (A.1)

Consider FE spaces (V_h, Q_h) satisfying a discrete inf-sup-condition:

$$\inf_{q\in Q_h\setminus\{0\}} \sup_{\mathbf{v}\in V_h\setminus\{0\}} \frac{(\nabla\cdot\mathbf{v}, q)}{\|\nabla\mathbf{v}\|_{L^2(\Omega)}\|q\|_{L^2(\Omega)}} \geq \beta > 0$$

$$\Rightarrow \quad {V_h}^{div} := \{\mathbf{v}_h \in V_h \mid (\nabla\cdot\mathbf{v}_h, q_h) = 0 \ \ \forall q_h \in Q_h\} \neq \{0\}$$

Assumption (A.2)

The fluctuation operator $\kappa_M = id - \pi_M$ provides the approximation property (depending on D_M and $s \in \{0, \cdots, k\}$):

$$\|\kappa_M \mathbf{w}\|_{L^2(M)} \leq C h_M^l \|\mathbf{w}\|_{W^{l,2}(M)}, \ \forall \mathbf{w} \in W^{l,2}(M), \ M \in \mathcal{M}_h, \ l \leq s.$$

A sufficient condition for (A.2) is $\mathbb{P}_{s-1} \subset D_M$.

Assumptions

Assumption (A.3)

Let the FE space V_h satisfy the local inverse inequality

$$\|\nabla \mathbf{v}_h\|_{L^2(M)} \leq C h_M^{-1} \|\mathbf{v}_h\|_{L^2(M)} \quad \forall \mathbf{v}_h \in V_h, \; M \in \mathcal{M}_h.$$

Assumption (A.4)

There are (quasi-)interpolation operators $j_u \colon V \to V_h$ and $j_p \colon Q \to Q_h$ such that for all $M \in \mathcal{M}_h$, for all $\mathbf{w} \in V \cap [W^{l,2}(\Omega)]^d$ with $2 \leq l \leq k+1$:

$$\|\mathbf{w} - j_u \mathbf{w}\|_{L^2(M)} + h_M \|\nabla(\mathbf{w} - j_u \mathbf{w})\|_{L^2(M)} \leq C h_M^l \|\mathbf{w}\|_{W^{l,2}(\omega_M)}$$

and for all $q \in Q \cap H^l(M)$ with $2 \leq l \leq k$ on a suitable patch $\omega_M \supset M$:

$$\|q - j_p q\|_{L^2(M)} + h_M \|\nabla(q - j_p q)\|_{L^2(M)} \leq C h_M^l \|q\|_{W^{l,2}(\omega_M)}$$

$$\|\mathbf{v} - j_u \mathbf{v}\|_{L^\infty(M)} \leq C h_M |\mathbf{v}|_{W^{1,\infty}(M)} \quad \forall \mathbf{v} \in [W^{1,\infty}(M)]^d.$$

Stabilization Terms

- LPS Streamline upwind Petrov-Galerkin (SUPG)

$$s_h(\mathbf{w}_h;\mathbf{u},\mathbf{v}) := \sum_{M\in\mathcal{M}_h} \tau_M(\mathbf{w}_M)(\kappa_M((\mathbf{w}_M\cdot\nabla)\mathbf{u}),\kappa_M((\mathbf{w}_M\cdot\nabla)\mathbf{v}))_M$$

- grad-div

$$t_h(\mathbf{w}_h;\mathbf{u},\mathbf{v}) := \sum_{M\in\mathcal{M}_h} \gamma_M(\mathbf{w}_M)(\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{v})_M$$

Stability and Existence

Stabilized Problem

Find $\boldsymbol{\mathcal{U}}_h = (\mathbf{u}_h, p_h) : (0, T) \to {V_h}^{div} \times Q_h$, s.t. $\forall \boldsymbol{\mathcal{V}}_h = (\mathbf{v}_h, q_h) \in {V_h}^{div} \times Q_h$

$$(\partial_t \mathbf{u}_h, \mathbf{v}_h) + A_G(\mathbf{u}_h; \boldsymbol{\mathcal{U}}_h, \boldsymbol{\mathcal{V}}_h) + (s_h + t_h)(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)$$

Stability

Define for $\boldsymbol{\mathcal{V}} \in V \times Q$ the norm $|||\boldsymbol{\mathcal{V}}|||_{LPS}^2 := \nu \|\nabla \mathbf{v}\|^2 + (s_h + t_h)(\boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{V}})$. Then the following stability result holds:

$$\|\mathbf{u}_h(t)\|_{L^2(\Omega)}^2 + \int_0^t |||\boldsymbol{\mathcal{U}}_h(s)|||_{LPS}^2 \, ds \leq \|\mathbf{u}_h(0)\|_{L^2(\Omega)}^2 + 3\|\mathbf{f}\|_{L^2(0,T;L^2(\Omega))}^2$$

Corollary (using the generalized Peano theorem)

$\exists$ discrete solution $\mathbf{u}_h : [0, T] \to {V_h}^{div}$ for the LPS model.

Theorem (A., D., Lube 2014)

Assume a solution according to

$$\mathbf{u} \in [L^\infty(0,T;W^{1,\infty}(\Omega)) \cap L^2(0,T;[W^{k+1,2}(\Omega))]^d,$$
$$\partial_t \mathbf{u} \in [L^2(0,T;W^{k,2}(\Omega))]^d,\ \ p \in L^2(0,T;W^{k,2}(\Omega)).$$

Then we obtain for $\mathbf{e}_h = \mathbf{u_h} - j_u \mathbf{u}$:

$$\|\mathbf{e}_h\|^2_{L^\infty(0,t);L^2(\Omega))} + \int_0^t |||\mathbf{e}_h(\tau)|||^2_{LPS}\, d\tau$$
$$\le C \sum_M h_M^{2k} \int_0^t e^{C_G(\mathbf{u})(t-\tau)} \Big[\min\Big(\frac{d}{\nu}, \frac{1}{\gamma_M}\Big) |p(\tau)|^2_{W^{k,2}(\omega_M)}$$
$$+ (1 + \nu Re_M^2 + \tau_M |\mathbf{u}_M|^2 + d\gamma_M) |\mathbf{u}(\tau)|^2_{W^{k+1,2}(\omega_M)}$$
$$+ \tau_M |\mathbf{u}_M|^2 h_M^{2(s-k)} |\mathbf{u}(\tau)|^2_{W^{s+1,2}(\omega_M)} + |\partial_t \mathbf{u}(\tau)|^2_{W^{k,2}(\omega_M)} \Big]\, d\tau$$

with $Re_M := \frac{h_M \|\mathbf{u}\|_{L^\infty(M)}}{\nu}$, $s \in \{0, \cdots, k\}$ and the Gronwall constant $C_G(\mathbf{u}) = 1 + C|\mathbf{u}|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + Ch\|\mathbf{u}\|^2_{L^\infty(0,T;W^{1,\infty}(\Omega))}$

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$$\begin{aligned}
&\|\mathbf{e}_h\|^2_{L^\infty(0,t);L^2(\Omega))} + \int_0^t |||\mathbf{e}_h(\tau)|||^2_{LPS} d\tau \\
&\leq C \sum_M h_M^{2k} \int_0^t e^{C_G(\mathbf{u})(t-\tau)} \Big[\min\Big(\frac{d}{\nu}, \frac{1}{\gamma_M}\Big) |p(\tau)|^2_{W^{k,2}(\omega_M)} \\
&\quad + (1 + \nu Re_M^2 + \tau_M|\mathbf{u}_M|^2 + d\gamma_M) |\mathbf{u}(\tau)|^2_{W^{k+1,2}(\omega_M)} \\
&\quad + \tau_M |\mathbf{u}_M|^2 h_M^{2(s-k)} |\mathbf{u}(\tau)|^2_{W^{s+1,2}(\omega_M)} + |\partial_t \mathbf{u}(\tau)|^2_{W^{k,2}(\omega_M)} \Big] d\tau
\end{aligned}$$

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Choice of Parameters and Projection Spaces

We achieve a method of order k provided

$$\nu Re_M^2 \leq C \Rightarrow h_M \leq C \frac{\sqrt{\nu}}{\|\mathbf{u}\|_{L^\infty(M)}}$$

$$\tau_M |\mathbf{u}_M|^2 h_M^{2(s-k)} \leq C \Rightarrow \tau_M \leq \tau_0 \frac{h_M^{2(k-s)}}{|\mathbf{u}_M|^2}$$

$$\max\{\frac{1}{\gamma_M}, \gamma_M\} \leq C \Rightarrow \gamma_M = \gamma_0$$

Examples for suitable projection spaces

- One-Level: $\mathbb{Q}_k/\mathbb{Q}_{k-1}/\mathbb{Q}_t$, $\mathbb{P}_k/\mathbb{P}_{k-1}/\mathbb{P}_t$, $\mathbb{Q}_k/\mathbb{P}_{-(k-1)}/\mathbb{P}_t \quad \forall t \leq k-1$
- Two-Level: $\mathbb{P}_k/\mathbb{P}_{k-1}/\mathbb{P}_t$, $\mathbb{Q}_k/\mathbb{Q}_{k-1}/\mathbb{Q}_t$, $\mathbb{Q}_k/\mathbb{P}_{-(k-1)}/\mathbb{P}_t \quad \forall t \leq k-1$

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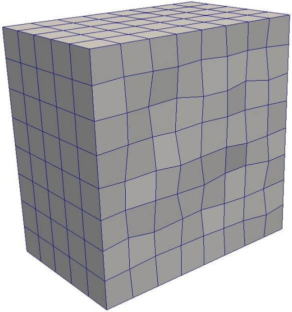
Couzy Testcase

Choose **f** such that the following pair is a solution in $\Omega = [0,1]^2$:

$$\mathbf{u}(\mathbf{x}) = \sin(\pi t) \begin{pmatrix} \cos\left(\frac{\pi}{2}x\right) \sin\left(\frac{\pi}{2}y\right) \sin\left(-\pi z\right) \\ \sin\left(\frac{\pi}{2}x\right) \cos\left(\frac{\pi}{2}y\right) \sin\left(-\pi z\right) \\ \sin\left(\frac{\pi}{2}x\right) \sin\left(\frac{\pi}{2}y\right) \cos\left(-\pi z\right) \end{pmatrix}$$

$$p(\mathbf{x}) = -\pi \sin\left(\frac{\pi}{2}x\right) \sin\left(\frac{\pi}{2}y\right) \sin\left(\frac{\pi}{2}x\right) \sin\left(-\pi z\right) \sin\left(\pi t\right)$$



Couzy Testcase

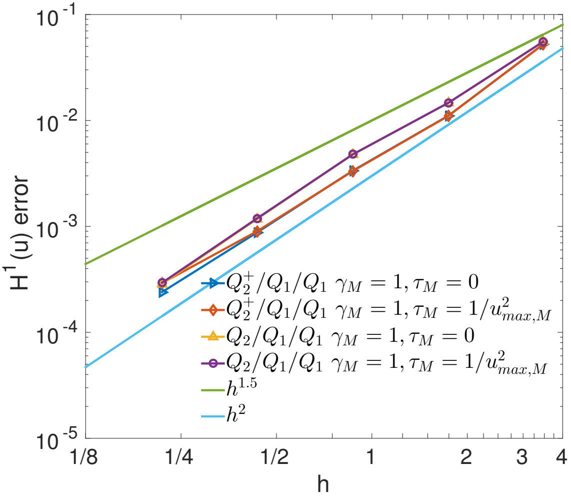


Figure: Errors for $Re = 10^3$ using periodic boundary conditions

Couzy Testcase

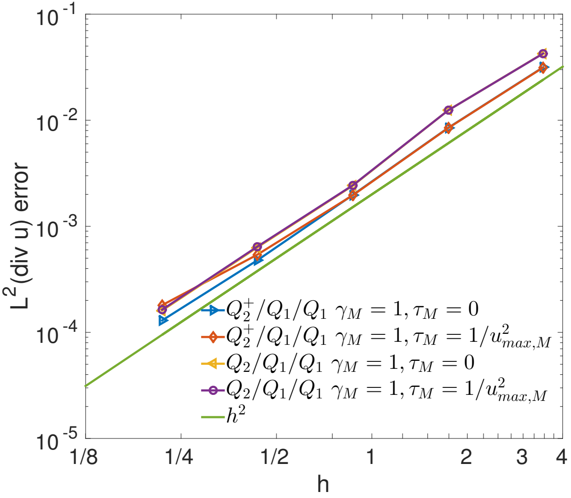


Figure: Errors for $Re = 10^3$ using periodic boundary conditions

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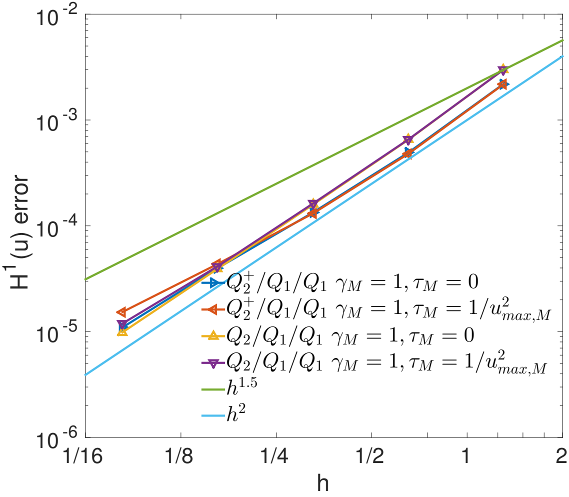


Figure: Errors for $Re = 10^3$ using inhomogeneous Dirichlet boundary data

Couzy Testcase

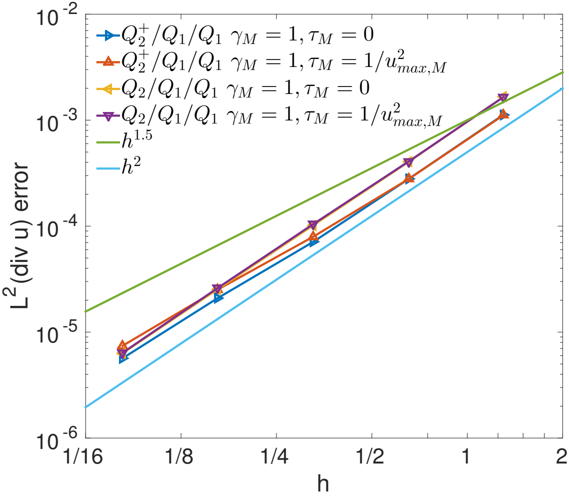


Figure: Errors for $Re = 10^3$ using inhomogeneous Dirichlet boundary data

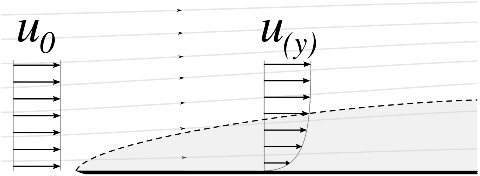
Overview

Blasius Flow, Boundary Layer Equation

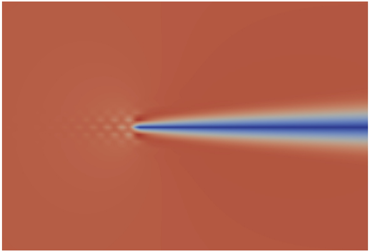
Consider the flow over an infinitesimal thin horizontal plate. Then the Navier-Stokes equations simplify to Prandtl's boundary layer equations:

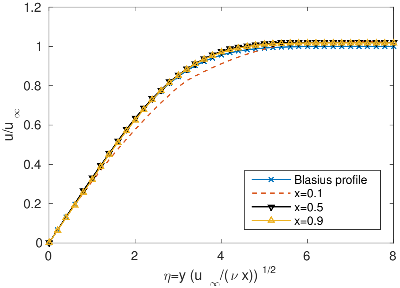
$$
\begin{aligned}
2f'''(\eta) + f(\eta)f''(\eta) &= 0 \\
f(0) &= 0 \\
f'(0) &= 0 \\
\lim_{x\to\infty} f'(\eta) &= 1
\end{aligned}
$$

$$\eta = y\sqrt{\frac{u_0}{2\nu x}}$$



Blasius Flow, $\nu = 10^{-3}$





$$Re_\Omega = 10^3, \gamma = 1, \tau_M = 0, h = 2^{-5}$$

Choice of the Stabilization Parameter τ_M

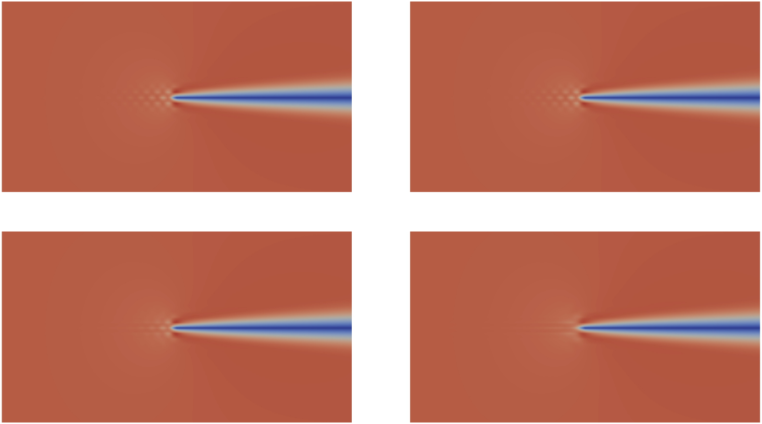


Abb: **a)** $\tau_M = 0$, **b)** $\tau_M = h^2/|\mathbf{u_M}|^2$, **c)** $\tau_M = h/|\mathbf{u_M}|^2$, **d)** $\tau_M = 1/|\mathbf{u_M}|^2$

Choice of the Coarse Space

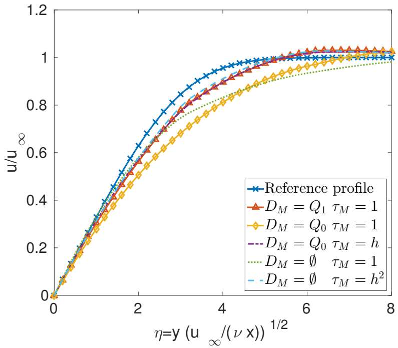
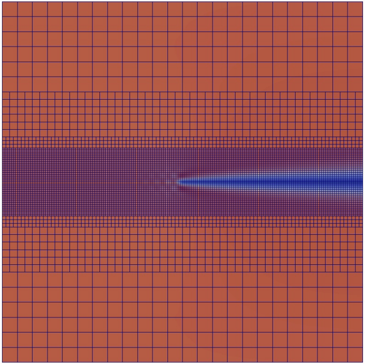


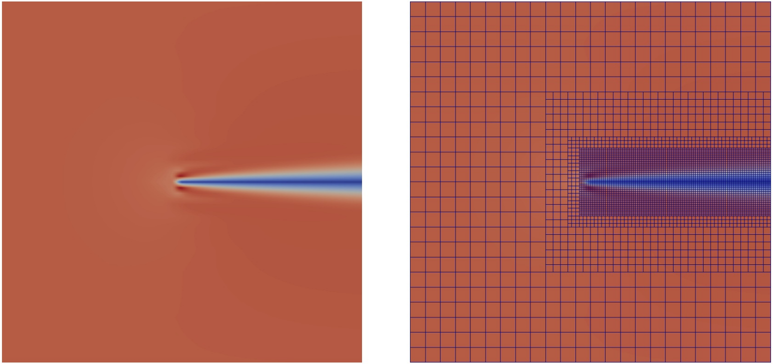
Figure: Profiles for various coarse spaces D_M and τ_M at $x = 0.1$ (left)
Profile $\mathbf{u}$ for $D_M = \emptyset$ and $\tau_M = 1$ (right)

Blasius Flow, $\nu = 10^{-3}$, grad-div



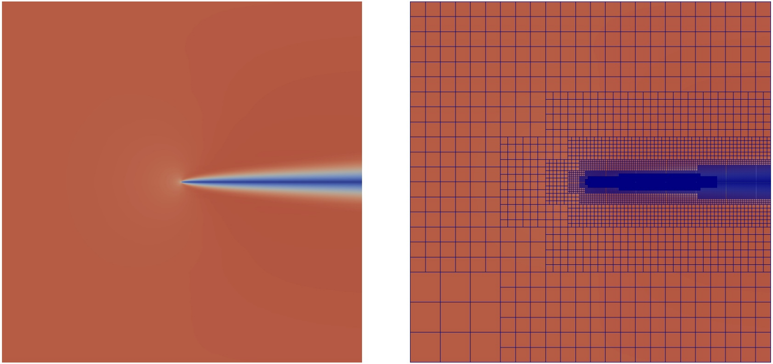
Refine cell K with midpoint (x, y) if $|y| < \delta$

Blasius Flow, $\nu = 10^{-3}$, grad-div



Refine in the boundary layer

Blasius Flow, $\nu = 10^{-3}$, grad-div

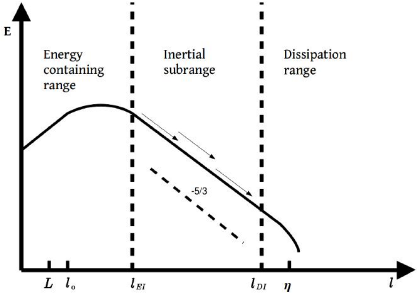


Refine cell K if $u_{max,K} - u_{min,K} > 1/5$

Overview

Energy Cascade

$$E(k,t) = \frac{1}{2} \sum_{k-\frac{1}{2} \leq \mathbf{k} \leq k+\frac{1}{2}} \hat{\mathbf{u}}(\mathbf{k},t) \cdot \hat{\mathbf{u}}(\mathbf{k},t)$$



Kolmogorov's second hypothesis

In the inertial subrange the energy is distributed like

$$E(k,t) = \alpha \varepsilon^{2/3} k^{-5/3}$$

assuming locally isotropic turbulence.

Taylor Green Vortex - Setting

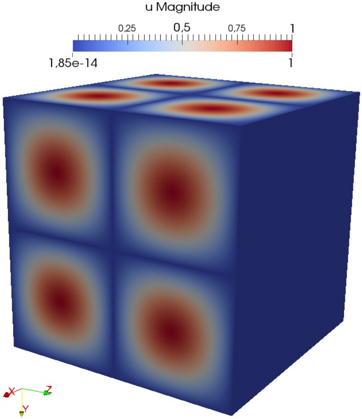
Flow in a perodic box $[0, 2\pi]^3$ for $Re = 10^4$ and initial data

$$\mathbf{u}_0 = \begin{pmatrix} \cos(x)\sin(y)\sin(z) \\ -\sin(x)\cos(y)\sin(z) \\ 0 \end{pmatrix}$$

$$p_0 = \frac{1}{16}\left(\cos(2x) + \cos(2y)\right)\left(\cos(2z) + 2\right).$$

The initial energy is concentrated on the wave numbers $\mathbf{k} = (\pm 1, \pm 1, \pm 1)$

$$E_0 = \pi^3 \mathbb{1}_{k=2}$$



Parameter Choice due to the Lilly argument

$$\tau_M(\kappa(\mathbf{u}_M \cdot \nabla \mathbf{u}_h), \kappa(\mathbf{u}_M \cdot \nabla \mathbf{v}_h)) \quad \tau_M = \widetilde{\tau_M} \frac{h^\beta}{\|\mathbf{u}_M\|^\gamma}$$

Assumption for the Lilly argument

$$\varepsilon = \tau_M \|\kappa(\mathbf{u}_M \cdot \nabla \mathbf{u}_h)\|^2 \quad E(k,t) = K_0 \varepsilon^{2/3} k^{-5/3}$$

$$\begin{aligned}
\|\kappa \nabla \mathbf{u}_h\|^2 &= \int_{k_c}^{k_f} k^2 E(k)\, dk = \alpha \int_{k_c}^{k_f} k^{1/3} \varepsilon^{2/3}\, dk \\
&= \alpha \int_{k_c}^{k_f} \tau_M^{2/3} \|\kappa(\mathbf{u}_M \cdot \nabla \mathbf{u}_h)\|^{4/3} k^{1/3}\, dk \\
&= C \frac{h^{2/3\beta}}{\|\mathbf{u}_M\|^{2/3\gamma}} \widetilde{\tau_M}^{2/3} \|\kappa(\mathbf{u}_M \cdot \nabla \mathbf{u}_h)\|^{4/3} h^{-4/3}
\end{aligned}$$

Parameter Choice due to the Lilly argument

$$\|\kappa \nabla \mathbf{u}_h\|^2 = C \frac{h^{2/3\beta}}{\|\mathbf{u}_M\|^{2/3\gamma}} \widetilde{\tau_M}^{2/3} \|\kappa(\mathbf{u}_M \cdot \nabla \mathbf{u}_h)\|^{4/3} h^{-4/3}$$

$$\Rightarrow \widetilde{\tau_M} = C \left(\frac{\|\kappa \nabla \mathbf{u}_h\|^2 \|\mathbf{u}_M\|^{2/3\gamma}}{h^{2/3\beta - 4/3} \|\kappa(\mathbf{u}_M \cdot \nabla \mathbf{u}_h)\|^{4/3}} \right)^{3/2} \overset{!}{=} const.$$

$$\geq C \left(\frac{\|\kappa \nabla \mathbf{u}_h\|^{2-4/3} \|\mathbf{u}_M\|^{2/3\gamma}}{h^{2/3\beta - 4/3} \|\mathbf{u}_M\|^{4/3}} \right)^{3/2}$$

$$\geq \|\nabla \mathbf{u}_h\| \|\mathbf{u}_M\|^{\gamma-2} h^{2-\beta}$$

$$\approx \|\mathbf{u}_M\|^{\gamma-1} h^{1-\beta}$$

Therefore $\gamma = \beta = 1$

Parameter Choice due to the Lilly argument

The LPS-SU stabilization has to satisfy $\tau_M \geq c\frac{h}{\|\mathbf{u}_M\|}$

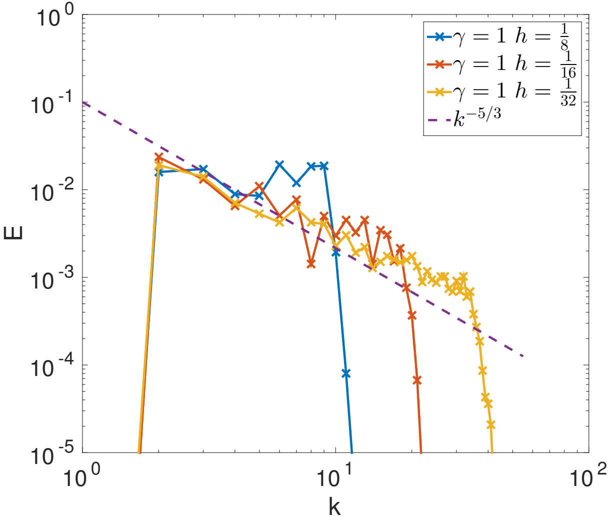
$$D_M = \mathbb{Q}^s$$

$$s = k \Rightarrow c\frac{h}{\|\mathbf{u}_M\|} \leq \tau_M \leq \frac{C}{\|\mathbf{u}_M\|^2}$$

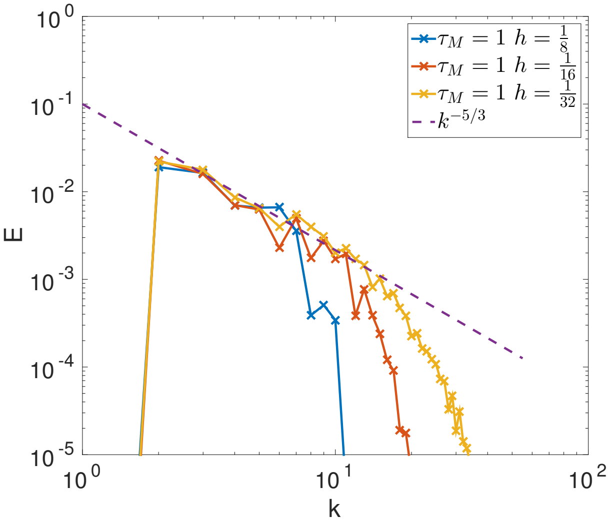
$$s = k-1 \Rightarrow c\frac{h}{\|\mathbf{u}_M\|} \leq \tau_M \leq C\frac{h}{\|\mathbf{u}_M\|^2}$$

$$s < k-1 \Rightarrow c\frac{h}{\|\mathbf{u}_M\|} \leq \tau_M \leq C\frac{h^{k-s}}{\|\mathbf{u}_M\|^2} \leq C\frac{h^2}{\|\mathbf{u}_M\|^2}$$

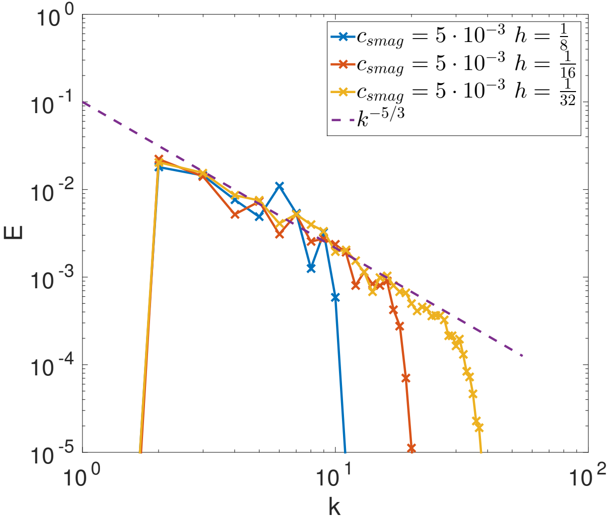
Energy Spectrum at $t = 9$, grad-div



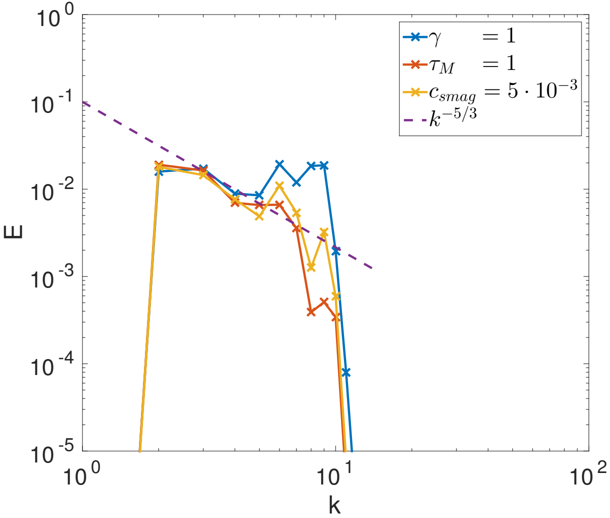
Energy Spectrum at $t = 9$, grad-div, LPS-SU



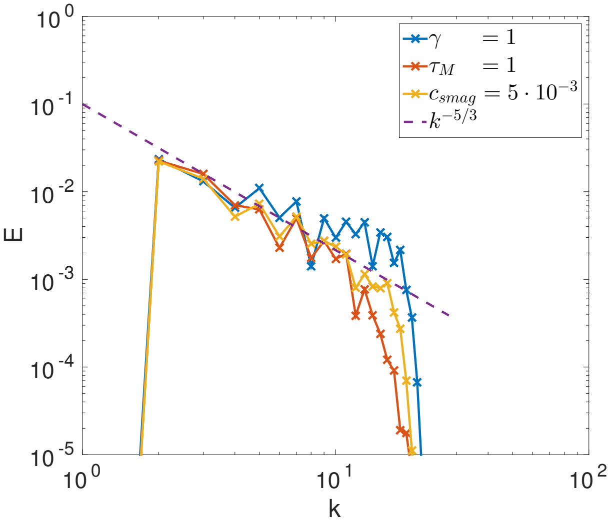
Energy Spectrum at $t = 9$, grad-div, Smagorinsky



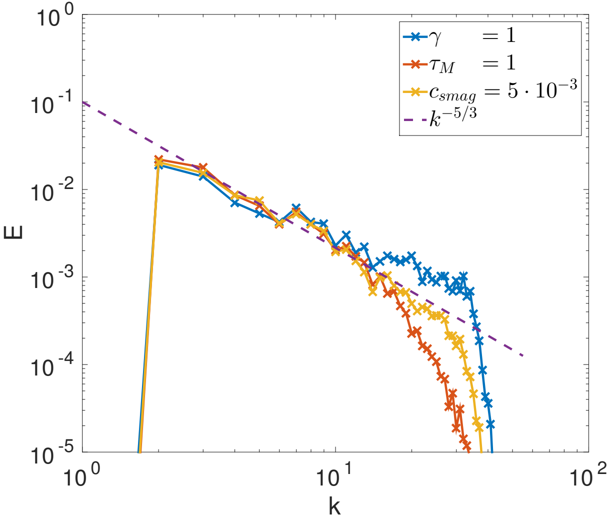
Energy Spectrum at $t = 9$, $h = 8^{-1}$



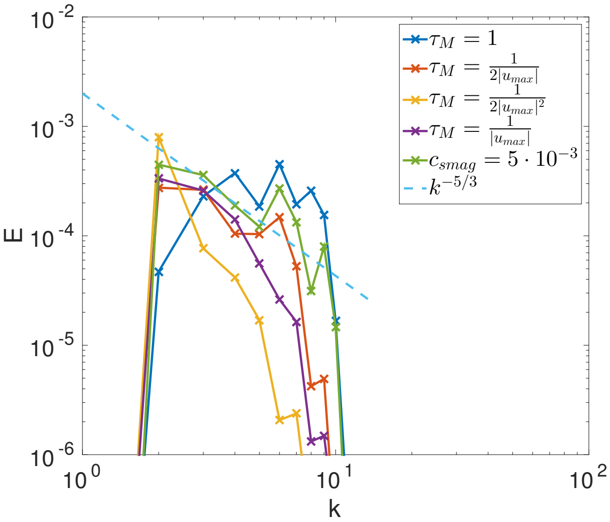
Energy Spectrum at $t = 9$, $h = 16^{-1}$



Energy Spectrum at $t = 9$, $h = 32^{-1}$



Energy Spectrum at $t = 9$, $h = 8^{-1}$, $\|\mathbf{u}_0\| = 10$



Overview

Summary

The Local Projection Stabilization approach provides

- Stability and Existence
- Quasi-optimal error estimates for standard discretizations (e.g. Taylor-Hood)

Numerical results confirm analytical estimates:

- For analytical examples SU stabilization is not necessary, but does not hurt either
- For non-convex domain SU stabilization prevents development of unphysical oszillations, parameters in the maximal possible range show best results
- We achieve satisfying results for isotropic turbulence, comparable to Smagorinsky. Improvement of parameter bounds by the Lilly argument.

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Thank you for your attention!