Error Estimates for the Fully Discretized Incompressible Navier-Stokes Problem with LPS Stabilization

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Abstract

We consider two different approaches for error estimates of the fully discretized time-dependent Navier-Stokes problem. For the spatial approximation we use conforming Finite Element Methods in conjunction with grad-div and local projection stabilization with respect to a streamline upwind/Petrov-Galerkin approach. For the temporal discretization a pressure correction projection algorithm based on BDF2 is used. In both cases we consider two semi-discretized problems. In the first strategy we discretize in time first and afterwards consider the error due to introducing stabilized Finite Element approximations in space. The second approach is the other way around. In both cases we can show (quasi-)optimal rates of convergence with respect to time discretization for all considered errors. With respect to spatially discretization the results suffer from a suboptimal estimate of the $L^2(\Omega)$ error in the velocity.

Contents

Chapter 1

Introduction

We consider the time-dependent Navier-Stokes equations

$$\begin{aligned} \partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad &\text{in} \ \ (t_0, T) \times \Omega, \\ \nabla \cdot \boldsymbol{u} = 0 \quad &\text{in} \ \ (t_0, T) \times \Omega, \\ \boldsymbol{u} = \boldsymbol{0} \quad &\text{in} \ \ (t_0, T) \times \partial\Omega, \\ \boldsymbol{u}(0, \cdot) = \boldsymbol{u}_0(\cdot) \quad &\text{in} \ \ \Omega \end{aligned} \tag{1.0.1}$$

in a bounded polyhedral domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$. Here $\boldsymbol{u} \colon (t_0, T) \times \Omega \to \mathbb{R}^d$ and $p \colon (t_0, T) \times \Omega \to \mathbb{R}$ denote the unknown velocity and pressure fields for given viscosity $\nu > 0$ and external forces $\boldsymbol{f} \in [L^2(t_0, T; [L^2(\Omega)]^d) \cap C(t_0, T; [L^2(\Omega)]^d)]^d$.

For the discretization with respect to time we use a splitting method called (standard) incremental pressure-correction projection method which is based on the backward differentiation formula of second order (BDF2). In the continuous problem $\boldsymbol{u}$ and p are coupled through the incompressibility constraint. The idea for pressure-correction projection methods is to define an auxiliary variable $\widetilde{\boldsymbol{u}}$ and solve for $\widetilde{\boldsymbol{u}}$ and p in two different steps such that the original velocity can be recovered form these two quantities. Such an approach was first considered by Chorin [1] and Temam [2]. An overview over different projection methods is given in [3] an overview over these methods is given. The incremental pressure correction algorithm with BDF2 time discretization is discussed by Guermond in [4] for the unstabilized Navier-Stokes equations and Shen considered a different second order time discretization scheme in [5]. It turns out that this technique suffers from unphysical boundary conditions for the pressure that lead to reduced rates of convergence. To prevent this Timmermans proposed in [6] the rotational pressure-correction projection method that uses a divergence correction for the pressure. A thorough analysis for this has first been performed in [7] for the Stokes problem.

This report discusses the fully discrete algorithm and its behavior in terms of convergence in time and space for the Navier-Stokes equations with grad-div and LPS-SU stabilization.

Two approaches for proving convergence are considered. In Chapter 2 we first discretize in time and apply afterwards a finite element approach for the spatial approximation. In Chapter 3 we interchange the order of approximations and apply the time discretization to a semi-discretized model that we considered in previous papers [8, 9].

We comprehend our considerations with numerical examples in Chapter 4 and finally discuss the results in Chapter 5.

1.1 LPS Method for the Navier-Stokes Problem

In this section, we describe the model problem and the spatial semi-discretization based on inf-sup stable interpolation of velocity and pressure together with local projection stabilization.

1.1.1 Time-Dependent Navier-Stokes Problem

In the following, we will consider the usual Sobolev spaces $W^{m,p}(\Omega)$ with norm $\|\cdot\|_{W^{m,p}(\Omega)}, m \in \mathbb{N}_0, p \geq 1$. In particular, we have $L^p(\Omega) = W^{0,p}(\Omega)$. Moreover, the closed subspaces $W_0^{1,2}(\Omega)$, consisting of functions in $W^{1,2}(\Omega)$ with zero trace on $\partial\Omega$, and $L_0^2(\Omega)$, consisting of L^2-functions with zero mean in Ω, will be used. The inner product in $L^2(D)$ with $D \subseteq \Omega$ will be denoted by $(\cdot,\cdot)_D$. In case of $D = \Omega$ we omit the index.

The variational formulation of problem (1.0.1) reads:

Find $\boldsymbol{\mathcal{U}} = (\boldsymbol{u}, p) \in [L^2(t_0,T;V) \cap L^\infty(t_0,T;[L^2(\Omega)]^d)] \times L^2(t_0,T;Q)$ where $\boldsymbol{V} \times Q := [W_0^{1,2}(\Omega)]^d \times L_0^2(\Omega)$ such that

$$(\partial_t \boldsymbol{u}, \boldsymbol{v}) + A_G(\boldsymbol{u}; \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{V}}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{\mathcal{V}} = (\boldsymbol{v}, q) \in \boldsymbol{V} \times Q \tag{1.1.1}$$

with the Galerkin form

$$\begin{aligned} A_G(\boldsymbol{w}; \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{V}}) := & \underbrace{\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (p, \nabla\cdot\boldsymbol{v}) + (q, \nabla\cdot\boldsymbol{u})}_{=: a_G(\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{V}})} \\ & + \underbrace{\frac{1}{2}\left[((\boldsymbol{w}\cdot\nabla)\boldsymbol{u}, \boldsymbol{v}) - ((\boldsymbol{w}\cdot\nabla)\boldsymbol{v}, \boldsymbol{u})\right]}_{= c(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})}. \end{aligned} \tag{1.1.2}$$

The skew-symmetric form of the convective term c is chosen for conservation purposes. In this paper, we will additionally assume that the velocity field $\boldsymbol{u}$ belongs to $L^\infty(t_0,T;[W^{1,\infty}(\Omega)]^d)$ which ensures uniqueness of the solution.

1.1.2 Finite Element Spaces

For a simplex $T \in \mathcal{T}_h$ or a quadrilateral/hexahedron T in $\mathbb{R}^d$, let $\hat{T}$ be the reference unit simplex or the unit cube $(-1,1)^d$. The bijective reference mapping

$F_T : \hat{T} \to T$ is affine for simplices and multi-linear for quadrilaterals/hexahedra. Let $\hat{\mathbb{P}}_l$ and $\hat{\mathbb{Q}}_l$ with $l \in \mathbb{N}_0$ be the set of polynomials of degree $\le l$ and of polynomials of degree $\le l$ in each variable separately. Moreover, we set

$$\mathbb{R}_l(\hat{T}) := \begin{cases} \mathbb{P}_l(\hat{T}) & \text{on simplices } \hat{T} \\ \mathbb{Q}_l(\hat{T}) & \text{on quadrilaterals/hexahedra } \hat{T}. \end{cases}$$

Define

$$\begin{aligned} Y_{h,-l} &:= \{\boldsymbol{v}_h \in L^2(\Omega) : \ \boldsymbol{v}_h|_T \circ F_T \in \mathbb{R}_l(\hat{T}) \ \forall T \in \mathcal{T}_h\}, \\ Y_{h,l} &:= Y_{h,-l} \cap W^{1,2}(\Omega). \end{aligned}$$

For convenience, we write $\boldsymbol{V}_h = \mathbb{R}_k$ instead of $\boldsymbol{V}_h = [Y_{h,k}]^d \cap \boldsymbol{V}$ and $Q_h = \mathbb{R}_{\pm(k-1)}$ instead of $Q_h = Y_{h,\pm(k-1)} \cap Q$.

Assumption 1.1.1. Let $\boldsymbol{V}_h \subseteq [Y_{h,k}]^d \cap \boldsymbol{V}$ and $Q_h \subseteq Y_{h,-k-1} \cap Q$ be FE spaces satisfying a discrete inf-sup-condition

$$\inf_{q \in Q_h \setminus \{0\}} \sup_{\boldsymbol{v} \in \boldsymbol{V}_h \setminus \{\boldsymbol{0}\}} \frac{(\nabla \cdot \boldsymbol{v}, q)}{\|\nabla \boldsymbol{v}\|_0 \|q\|_0} \ge \beta > 0 \tag{1.1.3}$$

with a constant β independent on h.

1.1.3 Local Projection Stabilization

For a Galerkin approximation of problem (1.1.1)-(1.1.2) on an admissible partition $\mathcal{T}_h$ of the polyhedral domain Ω, consider finite dimensional spaces $\boldsymbol{V}_h \times Q_h \subset \boldsymbol{V} \times Q$. Then, the semi-discretized problem reads:
Find $\boldsymbol{\mathcal{U}}_h = (\boldsymbol{u}_h, p_h) \colon (t_0, T) \to \boldsymbol{V}_h \times Q_h$ such that for all $\boldsymbol{\mathcal{V}}_h = (\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times Q_h$:

$$(\partial_t \boldsymbol{u}_h, \boldsymbol{v}_h) + A_G(\boldsymbol{u}_h; \boldsymbol{\mathcal{U}}_h, \boldsymbol{\mathcal{V}}_h) = (\boldsymbol{f}, \boldsymbol{v}_h). \tag{1.1.4}$$

The semi-discrete Galerkin solution of problem (1.1.4) may suffer from spurious oscillations due to poor mass conservation or dominating advection. The idea of local projection stabilization (LPS) methods is to separate discrete function spaces into small and large scales and to add stabilization terms only on small scales.

Let $\{\mathcal{M}_h\}$ be a family of shape-regular macro decompositions of Ω into d-simplices, quadrilaterals ($d = 2$) or hexahedra ($d = 3$). In the one-level LPS-approach, one has $\mathcal{M}_h = \mathcal{T}_h$. In the two-level LPS-approach, the decomposition $\mathcal{T}_h$ is derived from $\mathcal{M}_h$ by barycentric refinement of d-simplices or regular (dyadic) refinement of quadrilaterals and hexahedra. We denote by h_T and h the diameter of cells $T \in \mathcal{T}_h$ and $M \in \mathcal{M}_h$. It holds $h_T \le h \le C h_T$ for all $T \subset M$ and $M \in \mathcal{M}_h$.

Assumption 1.1.2. Let the FE space $\boldsymbol{Y}_{h,k}$ satisfy the local inverse inequality

$$\|\nabla \boldsymbol{v}_h\|_{0,M} \le C h^{-1} \|\boldsymbol{v}_h\|_{0,M} \quad \forall \boldsymbol{v}_h \in \boldsymbol{Y}_{h,k}, \ M \in \mathcal{M}_h. \tag{1.1.5}$$

Assumption 1.1.3. There are (quasi-)interpolation operators $j_u\colon \boldsymbol{V} \to \boldsymbol{V}_h$ and $j_p\colon Q \to Q_h$ such that for all $M \in \mathcal{M}_h$, for all $\boldsymbol{w} \in \boldsymbol{V} \cap [W^{l,2}(\Omega)]^d$ with $1 \le l \le k_u + 1$:

$$\|\boldsymbol{w} - j_u\boldsymbol{w}\|_{0,M} + h\|\nabla(\boldsymbol{w} - j_u\boldsymbol{w})\|_{0,M} \le Ch^l\|\boldsymbol{w}\|_{W^{l,2}(\omega_M)} \tag{1.1.6}$$

and for all $q \in Q \cap H^l(M)$ with $1 \le l \le k_p + 1$:

$$\|q - j_pq\|_{0,M} + h\|\nabla(q - j_pq)\|_{0,M} \le Ch^l\|q\|_{W^{l,2}(\omega_M)}. \tag{1.1.7}$$

on a suitable patch $\omega_M \supset M$. Moreover, let

$$\|\boldsymbol{v} - j_u\boldsymbol{v}\|_{L^\infty(M)} \le Ch|\boldsymbol{v}|_{W^{1,\infty}(M)} \quad \forall \boldsymbol{v} \in [W^{1,\infty}(M)]^d.$$

Let $\boldsymbol{D_M} \subset [L^\infty(M)]^d$ denote a FE space on $M \in \mathcal{M}_h$ for $\boldsymbol{u}_h$. For each $M \in \mathcal{M}_h$, let $\pi_M\colon [L^2(M)]^d \to \boldsymbol{D_M}$ be the orthogonal L^2-projection. Moreover, we denote by $\kappa_M := id - \pi_M$ the so-called fluctuation operator.

Assumption 1.1.4. The fluctuation operator $\kappa_M = id - \pi_M$ provides the approximation property (depending on $\boldsymbol{D_M}$ and $s \in \{0, \cdots, k\}$):

$$\|\kappa_M\boldsymbol{w}\|_{0,M} \le Ch^l\|\boldsymbol{w}\|_{W^{l,2}(M)},\ \forall \boldsymbol{w} \in W^{l,2}(M),\ M \in \mathcal{M}_h,\ l = 0, \ldots, s. \tag{1.1.8}$$

A sufficient condition for Assumption 1.1.4 is $[\mathbb{P}_{s-1}]^d \subset \boldsymbol{D_M}$.

Definition 1.1.5. For each macro element $M \in \mathcal{M}_h$ define the element-wise averaged streamline direction $\boldsymbol{u}_M \in \mathbb{R}^d$ by

$$\boldsymbol{u}_M := \frac{1}{|M|}\int_M \boldsymbol{u}(x)\ dx. \tag{1.1.9}$$

This choice gives the estimates:

$$|\boldsymbol{u}_M| \le C\|\boldsymbol{u}\|_{L^\infty(M)}, \qquad \|\boldsymbol{u} - \boldsymbol{u}_M\|_{L^\infty(M)} \le Ch|\boldsymbol{u}|_{W^{1,\infty}(M)}. \tag{1.1.10}$$

Now we can formulate the semi-discrete LPS model:
Find $\boldsymbol{\mathcal{U}}_h = (\boldsymbol{u}_h, p_h)\colon (t_0, T) \to \boldsymbol{V}_h \times Q_h$, such that for all $\boldsymbol{\mathcal{V}}_h = (\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times Q_h$:

$$(\partial_t\boldsymbol{u}_h, \boldsymbol{v}_h) + A_G(\boldsymbol{u}_h; \boldsymbol{\mathcal{U}}_h, \boldsymbol{\mathcal{V}}_h) + s_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + t(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \tag{1.1.11}$$

with the streamline-upwind (SUPG)-type stabilization s_h and the grad-div stabilization t according to

$$s_h(\boldsymbol{w}_h, \boldsymbol{u}, \boldsymbol{y}_h, \boldsymbol{v}) := \sum_{M \in \mathcal{M}_h} \tau_M(\kappa_M((\boldsymbol{w}_M \cdot \nabla)\boldsymbol{u}), \kappa_M((\boldsymbol{y}_M \cdot \nabla)\boldsymbol{v}))_M \tag{1.1.12}$$

$$t(\boldsymbol{u}, \boldsymbol{v}) := \gamma(\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v}). \tag{1.1.13}$$

The set of stabilization parameters τ_M, γ has to be determined later on.

1.2 Time Discretization

For the discretization of the time interval $[t_0, T]$ we consider N equidistant time steps of size $\Delta t = (T - t_0)/N$ yielding the set $M_T = \{t_0, \dots, t_N = T\}$.
In order to abbreviate the discrete time derivative we define the operator D_t by

$$D_t \boldsymbol{u}^n := \frac{3\boldsymbol{u}^n - 4\boldsymbol{u}^{n-1} + \boldsymbol{u}^{n-2}}{2\Delta t}. \tag{1.2.1}$$

Defining $\boldsymbol{Y}_h := \boldsymbol{V}_h^{div} \oplus \nabla Q_h \subset [L^2(\Omega)]^d$ the fully discretized and stabilized scheme reads:

Find $\widetilde{\boldsymbol{u}}_{ht}^n \in \boldsymbol{V}_h, \boldsymbol{u}_{ht}^n \in \boldsymbol{Y}_h$ and $p_{ht}^n \in Q_h$

$$\begin{aligned}
&\left(\frac{3\widetilde{\boldsymbol{u}}_{ht}^n - 4\boldsymbol{u}_{ht}^{n-1} + \boldsymbol{u}_{ht}^{n-2}}{2\Delta t}, \boldsymbol{v}_h\right) + \nu(\nabla \widetilde{\boldsymbol{u}}_{ht}^n, \nabla \boldsymbol{v}_h) + c(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) \\
&\quad + \gamma(\nabla \cdot \widetilde{\boldsymbol{u}}_{ht}^n, \nabla \cdot \boldsymbol{v}_h) + s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) \\
&\quad = (\boldsymbol{f}^n, \boldsymbol{v}_h) + (p_{ht}^{n-1}, \nabla \cdot \boldsymbol{v}_h) \\
&\widetilde{\boldsymbol{u}}_{ht}^n|_{\partial\Omega} = 0
\end{aligned} \tag{1.2.2}$$

$$\begin{aligned}
\left(\frac{3\boldsymbol{u}_{ht}^n - 3\widetilde{\boldsymbol{u}}_{ht}^n}{2\Delta t} + \nabla(p_{ht}^n - p_{ht}^{n-1}), \boldsymbol{y}_h\right) &= 0 \\
(\nabla \cdot \boldsymbol{u}_{ht}^n, q_h) &= 0 \\
\boldsymbol{u}_{ht}^n|_{\partial\Omega} &= 0
\end{aligned} \tag{1.2.3}$$

holds for all $\boldsymbol{v}_h \in \boldsymbol{V}_h,\ \boldsymbol{y}_h \in \boldsymbol{Y}_h$ and $q_h \in Q_h$.

From here on we assume $Q_h \subset H^1(\Omega)$. We call (1.2.2) the convection-diffusion step and (1.2.3) the projection step.
The LPS stabilization used here is a discretized version of the previous one. In particular,

$$s_h(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{y}, \boldsymbol{v}) := \sum_{M \in \mathcal{M}_h} \tau_M^n(\kappa_M((\boldsymbol{w}_M \cdot \nabla)\boldsymbol{u}), \kappa_M((\boldsymbol{y}_M \cdot \nabla)\boldsymbol{v})),$$

with $\boldsymbol{w}_M = \frac{1}{|M|}\int_M \boldsymbol{w}(x)dx$ and a stabilization parameter that coincides with the semi-discrete, time-continuous one at all t_n as $\tau_M^n = \tau_M(t_n)$. We assume that s_h is linear in each argument; in particular, τ_M^n must not depend nonlinearly on the arguments of s_h.

Remark 1.2.1. By testing equation 1.2.3 with $\boldsymbol{w}_h \in \boldsymbol{V}_h^{div}$ we derive

$$(\boldsymbol{u}_{ht}^n - \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{w}_h) = 0 \quad \forall \boldsymbol{w}_h \in \boldsymbol{V}_h^{div}. \tag{1.2.4}$$

Hence, $\boldsymbol{u}_{ht}^n$ is the $L^2(\Omega)$ projection of $\widetilde{\boldsymbol{u}}_{ht}^n$ onto $\boldsymbol{V}_h^{div}$ and $\|\boldsymbol{u}_{ht}^n\| \leq \|\widetilde{\boldsymbol{u}}_{ht}^n\|$.

Choosing a slightly bigger ansatz space $\boldsymbol{Y}_h^1 := \boldsymbol{V}_h + \nabla Q_h$ instead of $\boldsymbol{Y}_h$ we can eliminate the weakly solenoidal field $\widetilde{\boldsymbol{u}}_{ht}^n$ and replace (1.2.2) by the equation

$$\begin{aligned}
&(D_t\widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) + \nu(\nabla\widetilde{\boldsymbol{u}}_{ht}^n, \nabla\boldsymbol{v}_h) + c(\widetilde{\boldsymbol{u}}_{ht}^n; \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h)\\
&\quad + s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) + \gamma(\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^n, \nabla\cdot\boldsymbol{v}_h)\\
&\quad = (\boldsymbol{f}^n, \boldsymbol{v}_h) + \left(\frac{7}{3}p_{ht}^{n-1} - \frac{5}{3}p_{ht}^{n-2} + \frac{1}{3}p_{ht}^{n-3}, \nabla\cdot\boldsymbol{v}_h\right)\\
&\widetilde{\boldsymbol{u}}_{ht}^n|_{\partial\Omega} = 0
\end{aligned} \tag{1.2.5}$$

and equation (1.2.3) by

$$\begin{aligned}
(\nabla(p_{ht}^n - p_{ht}^{n-1}), \nabla q_h) &= \left(\frac{3\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^n}{2\Delta t}, q_h\right)\\
(\boldsymbol{n}\cdot\nabla p_{ht}^n)|_{\partial\Omega} &= 0.
\end{aligned} \tag{1.2.6}$$

$\boldsymbol{u}_{ht}^n$ can then be recovered according to

$$\boldsymbol{u}_{ht}^n = \widetilde{\boldsymbol{u}}_{ht}^n - \nabla(p_{ht}^n - p_{ht}^{n-1}).$$

This is the approach that we are going to use in our implementation. The equivalence of the two formulations (1.2.2), (1.2.3) and (1.2.5), (1.2.6) has been considered by Guermond in [10] for a first order unstabilized projection scheme.

Remark 1.2.2. For the first time step we use a BDF1 instead of the BDF2 scheme. In particular, the convection-diffusion step and the projection in the fully discretized setting read

Find $\widetilde{\boldsymbol{u}}_{ht}^n \in \boldsymbol{V}_h, \boldsymbol{u}_{ht}^n \in \boldsymbol{V}_h^{div}$ and $p_{ht}^n \in Q_h$ such that

$$\begin{aligned}
&\left(\frac{\widetilde{\boldsymbol{u}}_{ht}^1 - \boldsymbol{u}_{ht}^0}{\Delta t}, \boldsymbol{v}_h\right) + \nu(\nabla\widetilde{\boldsymbol{u}}_{ht}^1, \nabla\boldsymbol{v}_h) + c(\widetilde{\boldsymbol{u}}_{ht}^1; \widetilde{\boldsymbol{u}}_{ht}^1, \boldsymbol{v}_h)\\
&\quad + s_h(\widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{u}}_{ht}^1, \boldsymbol{v}_h) + \gamma(\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^1, \nabla\cdot\boldsymbol{v}_h)\\
&\quad = (\boldsymbol{f}^1, \boldsymbol{v}_h) + (p_{ht}^0, \nabla\cdot\boldsymbol{v}_h),\\
&\widetilde{\boldsymbol{u}}_{ht}^1|_{\partial\Omega} = 0\\
&\left(\frac{\boldsymbol{u}_{ht}^1 - \widetilde{\boldsymbol{u}}_{ht}^1}{\Delta t} + \nabla(p_{ht}^1 - p_{ht}^0), \boldsymbol{y}_h\right) = 0, \quad (\nabla\cdot\boldsymbol{u}_{ht}^1, q_h) = 0\\
&\boldsymbol{u}_{ht}^1|_{\partial\Omega} = 0
\end{aligned} \tag{1.2.7}$$

holds for all $\boldsymbol{v}_h \in \boldsymbol{V}_h, \boldsymbol{y}_h \in \boldsymbol{Y}_h$ and $q_h \in Q_h$.

The initial values are chosen according to $\widetilde{\boldsymbol{u}}_{ht}^0 = \boldsymbol{u}_{ht}^0 = j_u\boldsymbol{u}(t_0)$ and $p_{ht}^0 = j_p p(t_0)$ using the interpolation operators defined in Assumption 1.1.3.

Definition 1.2.3. Consider sequences $\boldsymbol{u} = (\boldsymbol{u}^1, \ldots, \boldsymbol{u}^N) \in \boldsymbol{A}^N$ of vector-valued and $p = (p^1, \ldots, p^N) \in B^N$ of scalar-valued quantities, where $\boldsymbol{A}$ and B are normed spaces and $1 \leq n \leq N$. The norms we want to control are defined by

$$\|\boldsymbol{u}\|^2_{l^2(t_0,T;\boldsymbol{A})} := \Delta t \sum_{n=1}^{N} \|\boldsymbol{u}^n\|^2_{\boldsymbol{A}}, \quad \|p\|^2_{l^2(t_0,T;B)} := \Delta t \sum_{n=1}^{N} \|p^n\|^2_B,$$
$$\|\boldsymbol{u}\|_{l^\infty(t_0,T;\boldsymbol{A})} := \max_{1\leq n\leq N} \|\boldsymbol{u}^n\|_{\boldsymbol{A}}, \quad \|p\|_{l^\infty(t_0,T;B)} := \max_{1\leq n\leq N} \|p^n\|_B.$$

For quantities r that are continuous in time we identify r by its evaluation at the discrete points in time $(r(t_1), \ldots, r(t_N))^T$.

Chapter 2

Temporal-Spatial Discretization

In this approach we want to consider the error that appears when discretizing the temporal approximation in space. By the triangle inequality the total error is then bounded as

$$\|\boldsymbol{\mathcal{U}} - \boldsymbol{\mathcal{U}}_{ht}\| \leq \|\boldsymbol{\mathcal{U}} - \boldsymbol{\mathcal{U}}_t\| + \|\boldsymbol{\mathcal{U}}_t - \boldsymbol{\mathcal{U}}_{ht}\| \tag{2.0.1}$$

The assumptions that we impose for this approach are given by

Assumption 2.0.4. The temporal discretized quantities fulfill the regularity requirement for any $1 \leq n \leq N$

$$\begin{aligned} \boldsymbol{u}_t^n &\in [H^1(\Omega)]^d, \quad p_t^n \in H^1(\Omega) \\ \widetilde{\boldsymbol{u}}_{ht}^n &\in [H^1(\Omega)]^d, \quad p_{ht}^n \in H^1(\Omega). \end{aligned} \tag{2.0.2}$$

and we assume for the continuous solution

$$\begin{aligned} &\boldsymbol{u} \in L^\infty(t_0, T; [W^{k_u+1,2}(\Omega)]^d), \quad \partial_t \boldsymbol{u} \in L^\infty(t_0, T; [H^2(\Omega)]^d) \\ &p \in L^2(t_0, T; W^{k_p+1,2}(\Omega)) \cap H^2(t_0, T; H^1(\Omega)). \end{aligned}$$

2.1 Stability of the Fully Discretized Scheme

A first important ingredient for our analysis is to derive a stability result on the fully discretized problem.

Lemma 2.1.1. *There exists $C > 0$ such that for all $1 \leq m \leq N$ it holds the* *stability result*

$$\begin{aligned}
&\|\widetilde{\boldsymbol{u}}_{ht}^m\|_0^2 + (\Delta t)^2\|\nabla p_{ht}^m\|_0^2 \\
&\quad + \Delta t \sum_{n=2}^{m} \Big[\nu\|\nabla\widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 + \gamma\|\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 \\
&\qquad\qquad + \|\delta_{tt}\boldsymbol{u}_{ht}^n\|_0^2 + s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n)\Big] \\
&\leq \|\widetilde{\boldsymbol{u}}_{ht}^0\|_0^2 + \|\widetilde{\boldsymbol{u}}_{ht}^1\|_0^2 + \|2\boldsymbol{u}_{ht}^1 - \boldsymbol{u}_{ht}^0\|_0^2 + \frac{4}{3}(\Delta t)^2\|\nabla p_{ht}^0\|_0^2 \\
&\quad + \frac{4}{3}(\Delta t)^2\|\nabla p_{ht}^1\|_0^2 + \Delta t\sum_{n=2}^{m}\frac{C}{\nu}\|\boldsymbol{f}^n\|_{-1}^2.
\end{aligned} \tag{2.1.1}$$

Proof. Due to the projection step (1.2.3) tested with p_{ht}^{n-1} we have

$$\begin{aligned}
-(\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^n, p_{ht}^{n-1}) + \frac{2\Delta t}{3}(\Delta(p_{ht}^n - p_{ht}^{n-1}), p_{ht}^{n-1}) = 0 \\
\frac{\partial p_{ht}^n}{\partial \boldsymbol{n}}|_{\partial\Omega} = \frac{\partial p_{ht}^{n-1}}{\partial \boldsymbol{n}}|_{\partial\Omega}.
\end{aligned} \tag{2.1.2}$$

Testing with $4\Delta t\widetilde{\boldsymbol{u}}_{ht}^n$ in (1.2.2) yields due to skew-symmetry of c

$$\begin{aligned}
&2\left(3\widetilde{\boldsymbol{u}}_{ht}^n - 4\boldsymbol{u}_{ht}^{n-1} + \boldsymbol{u}_{ht}^{n-2}, \widetilde{\boldsymbol{u}}_{ht}^n\right) + 4\Delta t\nu(\nabla\widetilde{\boldsymbol{u}}_{ht}^n, \nabla\widetilde{\boldsymbol{u}}_{ht}^n) \\
&\quad + 4\Delta t\gamma(\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^n, \nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^n) + 4\Delta t s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n) \\
&= -4\Delta t c(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n) + 4\Delta t(\boldsymbol{f}^n, \widetilde{\boldsymbol{u}}_{ht}^n) - 4\Delta t(\nabla p_{ht}^{n-1}, \widetilde{\boldsymbol{u}}_{ht}^n) \\
&= 4\Delta t(\boldsymbol{f}^n, \widetilde{\boldsymbol{u}}_{ht}^n) - 4\Delta t(\nabla p_{ht}^{n-1}, \widetilde{\boldsymbol{u}}_{ht}^n)
\end{aligned} \tag{2.1.3}$$

Adding these two equations together then gives

$$\begin{aligned}
&2\left(3\widetilde{\boldsymbol{u}}_{ht}^n - 4\boldsymbol{u}_{ht}^{n-1} + \boldsymbol{u}_{ht}^{n-2}, \widetilde{\boldsymbol{u}}_{ht}^n\right) + 4\Delta t\nu\|\nabla\widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 + 4\Delta t\gamma\|\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 \\
&\quad + 4\Delta t s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n) + \frac{8}{3}(\Delta t)^2(\nabla(p_{ht}^n - p_{ht}^{n-1}), \nabla p_{ht}^{n-1}) \\
&= 4\Delta t(\boldsymbol{f}^n, \widetilde{\boldsymbol{u}}_{ht}^n) \\
&\leq 4\frac{\nu\Delta t}{4}\|\nabla\widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 + 4\Delta t\frac{C}{\nu}\|\boldsymbol{f}^n\|_{-1}^2
\end{aligned} \tag{2.1.4}$$

The first term on the left can be split (cf. (A.1.1)) according to

$$\begin{aligned}
&(2(3\widetilde{\boldsymbol{u}}_{ht}^n - 4\boldsymbol{u}_{ht}^{n-1} + \boldsymbol{u}_{ht}^{n-2}), \widetilde{\boldsymbol{u}}_{ht}^n) = I_1 + I_2 + I_3 \\
&I_1 := 3\|\widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 + 3\|\boldsymbol{u}_{ht}^n - \widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 - 3\|\boldsymbol{u}_{ht}^n\|_0^2 \\
&I_2 := 2(\widetilde{\boldsymbol{u}}_{ht}^n - \boldsymbol{u}_{ht}^n, 3\boldsymbol{u}_{ht}^n - 4\boldsymbol{u}_{ht}^{n-1} + \boldsymbol{u}_{ht}^{n-2}) \\
&I_3 := \|\boldsymbol{u}_{ht}^n\|_0^2 + \|2\boldsymbol{u}_{ht}^n - \boldsymbol{u}_{ht}^{n-1}\|_0^2 + \|\delta_{tt}\boldsymbol{u}_{ht}^n\|_0^2 \\
&\qquad - \|\boldsymbol{u}_{ht}^{n-1}\|_0^2 - \|2\boldsymbol{u}_{ht}^{n-1} - \boldsymbol{u}_{ht}^{n-2}\|_0^2
\end{aligned}$$

The second term vanishes

$$\frac{3}{4\Delta t}I_2 = (\nabla(p_{ht}^n - p_{ht}^{n-1}), 3\boldsymbol{u}_{ht}^n - 4\boldsymbol{u}_{ht}^{n-1} + \boldsymbol{u}_{ht}^{n-2})$$

$$= -(p_{ht}^n - p_{ht}^{n-1}, \nabla \cdot (3\boldsymbol{u}_{ht}^n - 4\boldsymbol{u}_{ht}^{n-1} + \boldsymbol{u}_{ht}^{n-2})) = 0$$

due to the fact that $\boldsymbol{u}_{ht}^n$ is weakly divergence-free.
Using the identity $(a-b,b) = \frac{1}{2}(\|a\|_0^2 - \|a-b\|_0^2 - \|b\|_0^2)$ we have

$$\begin{aligned}
&3\|\widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 + 3\|\boldsymbol{u}_{ht}^n - \widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 - 3\|\boldsymbol{u}_{ht}^n\|_0^2 \\
&\quad + \|\boldsymbol{u}_{ht}^n\|_0^2 + \|2\boldsymbol{u}_{ht}^n - \boldsymbol{u}_{ht}^{n-1}\|_0^2 + \|\delta_{tt}\boldsymbol{u}_{ht}^n\|_0^2 - \|\boldsymbol{u}_{ht}^{n-1}\|_0^2 \\
&\quad - \|2\boldsymbol{u}_{ht}^{n-1} - \boldsymbol{u}_{ht}^{n-2}\|_0^2 + 4\Delta t\nu\|\nabla\widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 \\
&\quad + 4\Delta t\gamma\|\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 + 4\Delta t s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n) \\
&\quad + \frac{4}{3}(\Delta t)^2\left(\|\nabla p_{ht}^n\|_0^2 - \|\nabla(p_{ht}^n - p_{ht}^{n-1})\|_0^2 - \|\nabla p_{ht}^{n-1}\|_0^2\right) \\
&= 4\Delta t(\boldsymbol{f}^n, \widetilde{\boldsymbol{u}}_{ht}^n) \le \nu\Delta t\|\nabla\widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 + \frac{C\Delta t}{\nu}\|\boldsymbol{f}^n\|_{-1}^2
\end{aligned} \tag{2.1.5}$$

We now use that $\boldsymbol{u}_{ht}^n$ is an $L^2(\Omega)$ projection of $\widetilde{\boldsymbol{u}}_{ht}^n$ and therefore $\|\boldsymbol{u}_{ht}^n\| \le \|\widetilde{\boldsymbol{u}}_{ht}^n\|$ (cf. Remark 1.2.1):

$$\begin{aligned}
\Rightarrow\quad &\|\widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 + 3\|\boldsymbol{u}_{ht}^n - \widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 + \|2\boldsymbol{u}_{ht}^n - \boldsymbol{u}_{ht}^{n-1}\|_0^2 + \frac{4}{3}(\Delta t)^2\|\nabla p_{ht}^n\|_0^2 \\
&\quad + 3\Delta t\nu\|\nabla\widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 + 4\Delta t\gamma\|\nabla\cdot\boldsymbol{u}_{ht}^n\|_0^2 + \|\delta_{tt}\boldsymbol{u}_{ht}^n\|_0^2 \\
&\quad + 4\Delta t s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n) \\
&\le 3\|\widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 + 3\|\boldsymbol{u}_{ht}^n - \widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 - 2\|\boldsymbol{u}_{ht}^n\|_0^2 + \|2\boldsymbol{u}_{ht}^n - \boldsymbol{u}_{ht}^{n-1}\|_0^2 \\
&\quad + \frac{4}{3}(\Delta t)^2\|\nabla p_{ht}^n\|_0^2 + 3\Delta t\nu\|\nabla\widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 + 4\Delta t\gamma\|\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 \\
&\quad + \|\delta_{tt}\boldsymbol{u}_{ht}^n\|_0^2 + 4\Delta t s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n) \\
&\le \|\widetilde{\boldsymbol{u}}_{ht}^{n-1}\|_0^2 + \|2\boldsymbol{u}_{ht}^{n-1} - \boldsymbol{u}_{ht}^{n-2}\|_0^2 + \frac{4}{3}(\Delta t)^2\|\nabla p_{ht}^{n-1}\|_0^2 \\
&\quad + \frac{4}{3}(\Delta t)^2\|\nabla(p_{ht}^n - p_{ht}^{n-1})\|_0^2 + \frac{C\Delta t}{\nu}\|\boldsymbol{f}^n\|_{-1}^2
\end{aligned} \tag{2.1.6}$$

The projection equation tested with $p_{ht}^n - p_{ht}^{n-1}$ yields

$$\begin{aligned}
\frac{2\Delta t}{3}\|\nabla(p_{ht}^n - p_{ht}^{n-1})\|_0^2 &= -(\boldsymbol{u}_{ht}^n - \widetilde{\boldsymbol{u}}_{ht}^n, \nabla(p_{ht}^n - p_{ht}^{n-1})) \\
&\le \|\boldsymbol{u}_{ht}^n - \widetilde{\boldsymbol{u}}_{ht}^n\|\|\nabla(p_{ht}^n - p_{ht}^{n-1})\| \\
\Rightarrow\quad \frac{4}{3}(\Delta t)^2\|\nabla(p_{ht}^n - p_{ht}^{n-1})\|_0^2 &\le 3\|\boldsymbol{u}_{ht}^n - \widetilde{\boldsymbol{u}}_{ht}^n\|_0^2
\end{aligned} \tag{2.1.7}$$

We insert this in the previous estimate

$$\begin{aligned}
&\|\widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 + \|2\boldsymbol{u}_{ht}^n - \boldsymbol{u}_{ht}^{n-1}\|_0^2 + \frac{4}{3}(\Delta t)^2\|\nabla p_{ht}^n\|_0^2 + 3\Delta t\nu\|\nabla\widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 \\
&\quad + 4\Delta t\gamma\|\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 + \|\delta_{tt}\boldsymbol{u}_{ht}^n\|_0^2 + 4\Delta t s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n) \\
&\le \|\widetilde{\boldsymbol{u}}_{ht}^{n-1}\|_0^2 + \|2\boldsymbol{u}_{ht}^{n-1} - \boldsymbol{u}_{ht}^{n-2}\|_0^2 + \frac{4}{3}(\Delta t)^2\|\nabla p_{ht}^{n-1}\|_0^2 + \frac{C\Delta t}{\nu}\|\boldsymbol{f}^n\|_{-1}^2
\end{aligned} \tag{2.1.8}$$

and sum from $n = 2$ to m:

$$\begin{aligned}
&\|\widetilde{\boldsymbol{u}}_{ht}^m\|_0^2 + \|2\boldsymbol{u}_{ht}^m - \boldsymbol{u}_{ht}^{m-1}\|_0^2 + \frac{4}{3}(\Delta t)^2\|\nabla p_{ht}^m\|_0^2 \\
&\quad + \sum_{n=2}^{m}\Big[3\Delta t\nu\|\nabla\widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 + 4\Delta t\gamma\|\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 + \|\delta_{tt}\boldsymbol{u}_{ht}^n\|_0^2 \\
&\qquad\qquad + 4\Delta t s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n)\Big] \qquad (2.1.9)\\
&\le \|\widetilde{\boldsymbol{u}}_{ht}^0\|_0^2 + \|\widetilde{\boldsymbol{u}}_{ht}^1\|_0^2 + \|2\boldsymbol{u}_{ht}^1 - \boldsymbol{u}_{ht}^0\|_0^2 + \frac{4}{3}(\Delta t)^2\|\nabla p_{ht}^0\|_0^2 \\
&\quad + \frac{4}{3}(\Delta t)^2\|\nabla p_{ht}^1\|_0^2 + \Delta t\sum_{n=2}^{m}\frac{C}{\nu}\|\boldsymbol{f}^n\|_{-1}^2
\end{aligned}$$

and thus the desired estimate since $s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n) \ge 0$. □

2.2 Time Discretization

For the time discretized velocity $\boldsymbol{\mathcal{W}}_t = (\boldsymbol{u}_t, p_t)$ we consider a grad-div stabilized, but spatially continuous problem:

Find $\widetilde{\boldsymbol{w}}_t^n \in \boldsymbol{V}, \boldsymbol{w}_t^n \in \boldsymbol{V}^{div}$ and $r_t^n \in Q$ such that

$$\begin{aligned}
&\left(\frac{3\widetilde{\boldsymbol{w}}_t^n - 4\boldsymbol{w}_t^{n-1} + \boldsymbol{w}_t^{n-2}}{2\Delta t}, \boldsymbol{v}\right) + \nu(\nabla\widetilde{\boldsymbol{w}}_t^n, \nabla\boldsymbol{v}) + \gamma(\nabla\cdot\widetilde{\boldsymbol{w}}_t^n, \nabla\cdot\boldsymbol{v}) \\
&\quad = (\boldsymbol{f}^n, \boldsymbol{v}) - (\nabla r_t^{n-1}, \boldsymbol{v}) - c(\boldsymbol{u}(t_n); \boldsymbol{u}(t_n), \boldsymbol{v}) \qquad (2.2.1)\\
&\widetilde{\boldsymbol{w}}_t^n|_{\partial\Omega} = 0
\end{aligned}$$

$$\begin{aligned}
\left(\frac{3\boldsymbol{w}_t^n - 3\widetilde{\boldsymbol{w}}_t^n}{2\Delta t} + \nabla(r_t^n - r_t^{n-1}), \boldsymbol{v}\right) &= 0 \\
(\nabla\cdot\boldsymbol{w}_t^n, q) &= 0 \qquad (2.2.2)\\
\boldsymbol{w}_t^n|_{\partial\Omega} &= 0
\end{aligned}$$

holds for all $\boldsymbol{v} \in \boldsymbol{V}$ and $q \in Q$.

Compared to the fully discrete case where we test the projection equation in $\boldsymbol{Y}_h$ we do not need an auxiliary space due to the Helmholtz decomposition

$$\boldsymbol{V} = \boldsymbol{V}^{div} \oplus \nabla Q.$$

Definition 2.2.1. We denote the errors for this semi-discretization by

$$\boldsymbol{\eta}_u^n := \boldsymbol{u}(t_n) - \boldsymbol{w}_t^n \quad \widetilde{\boldsymbol{\eta}}_u^n := \boldsymbol{u}(t_n) - \widetilde{\boldsymbol{w}}_t^n, \quad \eta_p^n := p(t_n) - r_t^n.$$

2.2.1 Initialization of the Time Discretized Scheme

We first need a bound for the errors induced by the initialization. The result is given by:

Lemma 2.2.2. *The initial errors due to time discretization can be bounded as*

$$\begin{aligned}\|\nabla\eta_p^2\|_0^2+\|\nabla\eta_p^1\|_0^2+\frac{\nu}{\Delta t}(\|\nabla\widetilde{\boldsymbol{\eta}}_u^2\|_0^2+\|\nabla\widetilde{\boldsymbol{\eta}}_u^1\|_0^2)\le C(\Delta t)^2\\ \|\boldsymbol{\eta}_u^1\|_0^2+\|\boldsymbol{\eta}_u^2\|_0^2+\|\widetilde{\boldsymbol{\eta}}_u^2\|_0^2+\|\widetilde{\boldsymbol{\eta}}_u^1\|_0^2\le C(\Delta t)^4.\end{aligned}\tag{2.2.3}$$

Proof. The error equation due to the convection-diffusion step reads:

$$\begin{aligned}&\left(\frac{\widetilde{\boldsymbol{\eta}}_u^1-\boldsymbol{\eta}_u^0}{\Delta t},\boldsymbol{v}\right)+\nu(\nabla\widetilde{\boldsymbol{\eta}}_u^1,\nabla\boldsymbol{v})\\&+\gamma(\nabla\cdot\widetilde{\boldsymbol{\eta}}_u^1,\nabla\cdot\boldsymbol{v})+(\nabla(p^1-r_t^0),\boldsymbol{v})=(\boldsymbol{R}^1,\boldsymbol{v})\quad\forall\boldsymbol{v}\in\boldsymbol{V}\end{aligned}\tag{2.2.4}$$

where the residuum $\boldsymbol{R}^1$ is defined by

$$(\boldsymbol{R}^1,\boldsymbol{v}):=\left(\frac{\boldsymbol{u}(t_1)-\boldsymbol{u}(t_0)}{\Delta t}-\partial_t u(t_1),\boldsymbol{v}\right).$$

Testing this equation with $\widetilde{\boldsymbol{\eta}}_u^1$ yields

$$\begin{aligned}&\|\widetilde{\boldsymbol{\eta}}_u^1\|_0^2+\nu\Delta t\|\nabla\widetilde{\boldsymbol{\eta}}_u^1\|_0^2+\gamma\Delta t\|\nabla\cdot\widetilde{\boldsymbol{\eta}}_u^1\|_0^2\\&\le(\|\boldsymbol{\eta}_u^0\|_0+\Delta t(\|\nabla(p^0-p^1)\|_0+\|\boldsymbol{R}^1\|_0))\|\widetilde{\boldsymbol{\eta}}_u^1\|_0\\&\le C(\Delta t)^4(\|\boldsymbol{u}\|^2_{W^{1,\infty}(t_0,T;L^2)}+\|p\|^2_{W^{1,\infty}(t_0,T;H^1)}).\end{aligned}\tag{2.2.5}$$

Here we used $\|\boldsymbol{R}^1\|_0, \|\nabla\delta_t p(t_1)\|_0\le\Delta t$ due to the regularity assumptions 2.0.4 and (generalized) Taylor expansion and that $\boldsymbol{\eta}_u^0$ vanishes.

Next we consider the error equation due to the projection step. For all $q\in Q$ it holds

$$\left(\frac{\boldsymbol{\eta}_u^1-\widetilde{\boldsymbol{\eta}}_u^1}{\Delta t},\nabla q\right)+(\nabla(p(t_1)-r_t^1),\nabla q)=(\nabla(p(t_1)-p(t_0)),\nabla q).\tag{2.2.6}$$

Choosing $q=p(t_1)-r_t^1$ we arrive at

$$\begin{aligned}\Delta t\|\nabla(p(t_1)-r_t^1)\|_0^2&\le\|\widetilde{\boldsymbol{\eta}}_u^1\|_0+\Delta t\|(\nabla(p(t_0)-p(t_1))\|_0)\|\nabla(p(t_1)-r_t^1)\|_0\\&\le C(\Delta t)^3(\|\boldsymbol{u}\|^2_{W^{1,\infty}(t_0,T;L^2)}+\|p\|^2_{W^{1,\infty}(t_0,T;H^1)})\end{aligned}$$

where we used that $\boldsymbol{\eta}_u$ is weakly solenoidal. Testing (2.2.6) with $\boldsymbol{\eta}_u^1$ gives

$$\|\boldsymbol{\eta}_u^1\|_0^2\le(\|\widetilde{\boldsymbol{\eta}}_u^1\|_0+\Delta t\|\nabla(p(t_0)-p(t_1)\|_0+\|\nabla(p(t_1)-r_t^1)\|_0)\|)\boldsymbol{\eta}_u^1\|_0\tag{2.2.7}$$

and finally $\|\boldsymbol{\eta}_u^1\|_0^2\le C(\Delta t)^4(\|\boldsymbol{u}\|^2_{W^{1,\infty}(t_0,T;L^2)}+\|p\|^2_{W^{1,\infty}(t_0,T;H^1)})$.

Next, we need an estimate for $\widetilde{\boldsymbol{\eta}}_u^2$. Applying the same technique for $n = 2$ with the abbreviation $\boldsymbol{R}^2 := D_t \boldsymbol{u}(t_2) - \partial_t \boldsymbol{u}(t_2)$ gives

$$
\begin{aligned}
&\left(\frac{3\widetilde{\boldsymbol{\eta}}_u^2 - 3\widetilde{\boldsymbol{\eta}}_u^1}{2\Delta t}, \widetilde{\boldsymbol{\eta}}_u^2\right) + \nu\left(\nabla\left(\widetilde{\boldsymbol{\eta}}_u^2 - \widetilde{\boldsymbol{\eta}}_u^1\right), \nabla\widetilde{\boldsymbol{\eta}}_u^2\right) \\
&= (\boldsymbol{R}^2, \widetilde{\boldsymbol{\eta}}_u^2) + \nabla\left(r_t^1 - p\left(t_2\right), \widetilde{\boldsymbol{\eta}}_u^2\right) - \nu\left(\nabla\left(\widetilde{\boldsymbol{\eta}}_u^1 - \widetilde{\boldsymbol{\eta}}_u^0\right), \nabla\widetilde{\boldsymbol{\eta}}_u^2\right) \\
&\quad + \left(\frac{3\boldsymbol{\eta}_u^1 - 3\widetilde{\boldsymbol{\eta}}_u^1}{2\Delta t}, \widetilde{\boldsymbol{\eta}}_u^2\right) + \left(\frac{\boldsymbol{\eta}_u^1 - \boldsymbol{\eta}_u^0}{2\Delta t}, \widetilde{\boldsymbol{\eta}}_u^2\right) \\
&= (\boldsymbol{R}^2, \widetilde{\boldsymbol{\eta}}_u^2) + \left(\nabla\left(r_t^1 - p\left(t_2\right)\right), \widetilde{\boldsymbol{\eta}}_u^2\right) - \nu\left(\nabla\left(\widetilde{\boldsymbol{\eta}}_u^1 - \widetilde{\boldsymbol{\eta}}_u^0\right), \nabla\widetilde{\boldsymbol{\eta}}_u^2\right) \\
&\quad + \left(\frac{\boldsymbol{\eta}_u^1 - \boldsymbol{\eta}_u^0}{2\Delta t}, \widetilde{\boldsymbol{\eta}}_u^2\right) + \frac{3}{2}\left(\nabla\left(r_t^1 - r_t^0\right), \widetilde{\boldsymbol{\eta}}_u^2\right) \\
&= (\boldsymbol{R}^2, \widetilde{\boldsymbol{\eta}}_u^2) - \nu\left(\nabla\left(\widetilde{\boldsymbol{\eta}}_u^1 - \widetilde{\boldsymbol{\eta}}_u^0\right), \nabla\widetilde{\boldsymbol{\eta}}_u^2\right) \\
&\quad + \left(\nabla\left(\frac{5}{2}r_t^1 - \frac{3}{2}r_t^0 - p\left(t_2\right)\right), \widetilde{\boldsymbol{\eta}}_u^2\right) + \left(\frac{\boldsymbol{\eta}_u^1 - \boldsymbol{\eta}_u^0}{2\Delta t}, \widetilde{\boldsymbol{\eta}}_u^2\right) \\
&= (\boldsymbol{R}^2, \widetilde{\boldsymbol{\eta}}_u^2) - \nu\left(\nabla\left(\widetilde{\boldsymbol{\eta}}_u^1 - \widetilde{\boldsymbol{\eta}}_u^0\right), \nabla\widetilde{\boldsymbol{\eta}}_u^2\right) + \left(\frac{\boldsymbol{\eta}_u^1 - \boldsymbol{\eta}_u^0}{2\Delta t}, \widetilde{\boldsymbol{\eta}}_u^2\right) \\
&\quad + \frac{5}{2}\left(\nabla\left(r_t^1 - p\left(t_1\right)\right), \widetilde{\boldsymbol{\eta}}_u^2\right) + \left(\nabla\left(\frac{5}{2}p\left(t_1\right) - \frac{3}{2}p\left(t_0\right) - p\left(t_2\right)\right), \widetilde{\boldsymbol{\eta}}_u^2\right) \\
&\leq \min\{C\Delta t\|\widetilde{\boldsymbol{\eta}}_u^2\|_0, C\Delta t\|\widetilde{\boldsymbol{\eta}}_u^2\|_1\}(\|\boldsymbol{u}\|^2_{W^{1,\infty}(t_0,T;L^2)} + \|p\|^2_{W^{1,\infty}(t_0,T;H^1)}) \\
\Rightarrow &\|\widetilde{\boldsymbol{\eta}}_u^2 - \widetilde{\boldsymbol{\eta}}_u^1\|_0^2 + \nu\Delta t\|\widetilde{\boldsymbol{\eta}}_u^2 - \widetilde{\boldsymbol{\eta}}_u^1\|_1^2 \\
&\leq C\left(\Delta t\right)^4\left(\|\boldsymbol{u}\|^2_{W^{1,\infty}(t_0,T;L^2)} + \|p\|^2_{W^{1,\infty}(t_0,T;H^1)}\right).
\end{aligned}
$$

Since the error $\boldsymbol{\eta}_u^1$ is an orthogonal $L^2(\Omega)$ projection of $\widetilde{\boldsymbol{\eta}}_u^1$ we also get

$$
\|\boldsymbol{\eta}_u^2 - \boldsymbol{\eta}_u^1\|_0^2 \leq \|\widetilde{\boldsymbol{\eta}}_u^2 - \widetilde{\boldsymbol{\eta}}_u^1\|_0^2 \leq C(\Delta t)^4(\|\boldsymbol{u}\|^2_{W^{1,\infty}(t_0,T;L^2)} + \|p\|^2_{W^{1,\infty}(t_0,T;H^1)})
$$

For the pressure error we again use the projection equation

$$
\begin{aligned}
(\nabla(\eta_p^1 - \eta_p^2), \nabla q) &= (\nabla(p(t_1) - p(t_2)), \nabla q) - \left(\frac{3\boldsymbol{\eta}_u^2 - 3\widetilde{\boldsymbol{\eta}}_u^2}{2\Delta t}, \nabla q\right) \\
&\leq C\Delta t\|\nabla q\|_0(\|\boldsymbol{u}\|_{W^{1,\infty}(t_0,T;L^2)} + \|p\|_{W^{1,\infty}(t_0,T;H^1)}) \qquad (2.2.8) \\
\Rightarrow \|\nabla\eta_p^2\|_0^2 &\leq \|\nabla\eta_p^1\|_0^2 + \|\nabla(\eta_p^1 - \eta_p^2)\|_0^2 \\
&\leq C(\Delta t)^2(\|\boldsymbol{u}\|^2_{W^{1,\infty}(t_0,T;L^2)} + \|p\|^2_{W^{1,\infty}(t_0,T;H^1)}).
\end{aligned}
$$

□

2.2.2 Error Estimates after Initialization

Now that we know how to bound the initial errors we first derive an estimate on the difference between the solenoidal error $\boldsymbol{\eta}_u^m$ and the non-solenoidal error

$\widetilde{\boldsymbol{\eta}}_u^m$. The derived estimates give us the convergence behavior with respect to the $H^1(\Omega)$ and the divergence error. Finally, an inverse Stokes operator is used to get the desired estimates for $\|\widetilde{\boldsymbol{\eta}}_u^m\|_0$.

Lemma 2.2.3. *For all $1 \leq m \leq N$ it holds*

$$\|\boldsymbol{\eta}_u^m - \widetilde{\boldsymbol{\eta}}_u^m\|_0 \leq C(\Delta t)^2.$$

Proof. The error equation due to the convection-diffusion step (2.2.1) reads

$$\begin{aligned}&\left(\frac{3\widetilde{\boldsymbol{\eta}}_u^n - 4\boldsymbol{\eta}_u^{n-1} + \boldsymbol{\eta}_u^{n-2}}{2\Delta t}, \boldsymbol{v}\right) + \nu(\nabla\widetilde{\boldsymbol{\eta}}_u^n, \nabla \boldsymbol{v}) + \gamma(\nabla \cdot \widetilde{\boldsymbol{\xi}}^n, \nabla \cdot \boldsymbol{v}) \\ &= (\boldsymbol{R}^n, \boldsymbol{v}) - (\nabla(p(t_n) - r_t^{n-1}), \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}\end{aligned} \tag{2.2.9}$$

where the residuum $\boldsymbol{R}^n$ is given by

$$(\boldsymbol{R}^n, \boldsymbol{v}) = \left(\frac{3\boldsymbol{u}(t_n) - 4\boldsymbol{u}(t_{n-1}) + \boldsymbol{u}(t_{n-2})}{2\Delta t} - \partial_t \boldsymbol{u}(t_n), \boldsymbol{v}\right).$$

Since the propagation operator is linear we also get

$$\begin{aligned}&\left(\frac{3\delta_t\widetilde{\boldsymbol{\eta}}_u^n - 4\delta_t\boldsymbol{\eta}_u^{n-1} + \delta_t\boldsymbol{\eta}_u^{n-2}}{2\Delta t}, \boldsymbol{v}\right) \\ &\quad + \nu(\nabla\delta_t\widetilde{\boldsymbol{\eta}}_u^n, \nabla \boldsymbol{v}) + \gamma(\nabla \cdot \delta_t\widetilde{\boldsymbol{\eta}}_u^n, \nabla \cdot \boldsymbol{v}) \\ &= (\delta_t\boldsymbol{R}^n, \boldsymbol{v}) - (\nabla\delta_t(p(t_n) - r_t^{n-1}), \boldsymbol{v}).\end{aligned} \tag{2.2.10}$$

Now, we can do the same for the error due to the projection step (2.2.2):

$$\left(\frac{3\delta_t\boldsymbol{\eta}_u^n - 3\delta_t\widetilde{\boldsymbol{\eta}}_u^n}{2\Delta t} + \nabla\delta_t\eta_p^n - \nabla\delta_t(p(t_n) - r_t^{n-1}), \nabla q\right) = 0. \tag{2.2.11}$$

Testing the incremental error equation (2.2.10) with $4\Delta t\delta_t\widetilde{\boldsymbol{\eta}}_u^n$ we arrive at

$$\begin{aligned}&(2(3\delta_t\widetilde{\boldsymbol{\eta}}_u^n - 4\delta_t\boldsymbol{\eta}_u^{n-1} + \delta_t\boldsymbol{\eta}_u^{n-2}), \delta_t\widetilde{\boldsymbol{\eta}}_u^n) \\ &\quad + 4\Delta t\nu\|\nabla\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + 4\Delta t\gamma\|\nabla \cdot \delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 \\ &= 4\Delta t(\delta_t\boldsymbol{R}^n - \nabla\delta_t(p(t_n) - r_t^{n-1}), \delta_t\widetilde{\boldsymbol{\eta}}_u^n).\end{aligned} \tag{2.2.12}$$

The first term is then split (cf. (A.1.1)) according to

$$\begin{aligned}&(2(3\delta_t\widetilde{\boldsymbol{\eta}}_u^n - 4\delta_t\boldsymbol{\eta}_u^{n-1} + \delta_t\boldsymbol{\eta}_u^{n-2}), \delta_t\widetilde{\boldsymbol{\eta}}_u^n) = I_1 + I_2 + I_3 \\ &I_1 := 3\|\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + 3\|\delta_t\boldsymbol{\eta}_u^n - \delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 - 3\|\delta_t\boldsymbol{\eta}_u^n\|_0^2 \\ &I_2 := 2(\delta_t\widetilde{\boldsymbol{\eta}}_u^n - \delta_t\boldsymbol{\eta}_u^n, 3\delta_t\boldsymbol{\eta}_u^n - 4\delta_t\boldsymbol{\eta}_u^{n-1} + \delta_t\boldsymbol{\eta}_u^{n-2}) \\ &I_3 := \|\delta_t\boldsymbol{\eta}_u^n\|_0^2 + \|2\delta_t\boldsymbol{\eta}_u^n - \delta_t\boldsymbol{\eta}_u^{n-1}\|_0^2 + \|\delta_{ttt}\boldsymbol{\eta}_u^n\|_0^2 \\ &\qquad - \|\delta_t\boldsymbol{\eta}_u^{n-1}\|_0^2 - \|2\delta_t\boldsymbol{\eta}_u^{n-1} - \delta_t\boldsymbol{\eta}_u^{n-2}\|_0^2\end{aligned}$$

The second term I_2 vanishes

$$
\begin{aligned}
\frac{3}{4\Delta t} I_2 &= (\nabla(\delta_t(p(t_n) - r_t^{n-1}) - \delta_t \eta_p^n), 3\delta_t \boldsymbol{\eta}_u^n - 4\delta_t \boldsymbol{\eta}_u^{n-1} + \delta_t \boldsymbol{\eta}_u^{n-2}) \\
&= -(\delta_t(p(t_n) - r_t^{n-1}) - \delta_t \eta_p^n, \nabla \cdot (3\delta_t \boldsymbol{\eta}_u^n - 4\delta_t \boldsymbol{\eta}_u^{n-1} + \delta_t \boldsymbol{\eta}_u^{n-2})) \\
&= 0
\end{aligned}
$$

due to the fact that $\boldsymbol{u}$ and $\boldsymbol{w}^n$ are weakly divergence-free.

Now, we test the error in the projection step (2.2.11) with $\delta_t \eta_p^{n-1} = \delta_t(p(t_{n-1}) - r_t^{n-1})$ and get after integration by parts for the first term

$$
\begin{aligned}
& -\left(\frac{3}{2\Delta t} \delta_t \widetilde{\boldsymbol{\eta}}_u^n, \nabla \delta_t (p(t_{n-1}) - r_t^{n-1})\right) \\
&= -(\nabla \delta_t(\eta_p^n - \eta_p^{n-1}), \nabla \delta_t \eta_p^{n-1}) + (\nabla \delta_t (p(t_n) - p(t_{n-1})), \nabla \delta_t \eta_p^{n-1}) \\
&= -\frac{1}{2}(\|\nabla \delta_t \eta_p^n\|_0^2 - \|\nabla \delta_t(\eta_p^n - \eta_p^{n-1})\|_0^2 - \|\nabla \delta_t \eta_p^{n-1}\|_0^2) \\
&\quad + (\nabla \delta_t (p(t_n) - p(t_{n-1})), \nabla \delta_t \eta_p^{n-1})
\end{aligned}
$$

Using the projection step (2.2.11) again and testing with $\delta_t(\eta_p^n - \eta_p^{n-1})$ we derive

$$
\begin{aligned}
& \frac{2\Delta t}{3} \|\nabla \delta_t (\eta_p^n - \eta_p^{n-1})\|_0^2 \\
& \leq \|\delta_t \boldsymbol{\eta}_u^n - \delta_t \widetilde{\boldsymbol{\eta}}_u^n\|_0 \|\nabla \delta_t (\eta_p^n - \eta_p^{n-1})\|_0 \\
& \quad + \frac{2\Delta t}{3} (\nabla \delta_t (p(t_n) - p(t_{n-1})), \nabla \delta_t (\eta_p^n - \eta_p^{n-1})) \\
& \leq \frac{3}{4\Delta t} \|\delta_t \boldsymbol{\eta}_u^n - \delta_t \widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + \frac{\Delta t}{3} \|\nabla \delta_t (\eta_p^n - \eta_p^{n-1})\|_0^2 \\
& \quad + \frac{2\Delta t}{3} (\nabla \delta_{tt} p(t_n), \nabla \delta_t (\eta_p^n - \eta_p^{n-1}))
\end{aligned} \tag{2.2.13}
$$

and thus

$$
\begin{aligned}
& -4\Delta t \left(\delta_t \widetilde{\boldsymbol{\eta}}_u^n, \nabla \delta_t (p(t_{n-1}) - r_t^{n-1})\right) \\
& \leq \frac{8}{3} (\Delta t)^2 (\nabla \delta_t (p(t_n) - p(t_{n-1})), \nabla \delta_t \eta_p^{n-1}) - \frac{4}{3} (\Delta t)^2 \|\nabla \delta_t \eta_p^n\|_0^2 \\
& \quad + \frac{4}{3} (\Delta t)^2 \|\nabla \delta_t \eta_p^{n-1}\|_0^2 + 3\|\delta_t \boldsymbol{\eta}_u^n - \delta_t \widetilde{\boldsymbol{\eta}}_u^n\|_0^2 \\
& \quad + \frac{8}{3} (\Delta t)^2 (\nabla \delta_{tt} p(t_n), \nabla \delta_t (\eta_p^n - \eta_p^{n-1})).
\end{aligned}
$$

Combining all the estimates gives

$$
\begin{aligned}
& 3\|\delta_t \widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + 3\|\delta_t \boldsymbol{\eta}_u^n - \delta_t \widetilde{\boldsymbol{\eta}}_u^n\|_0^2 - 3\|\delta_t \boldsymbol{\eta}_u^n\|_0^2 \\
& \quad + \|\delta_t \boldsymbol{\eta}_u^n\|_0^2 + \|2\delta_t \boldsymbol{\eta}_u^n - \delta_t \boldsymbol{\eta}_u^{n-1}\|_0^2 + \|\delta_{ttt} \boldsymbol{\eta}_u^n\|_0^2 \\
& \quad - \|\delta_t \boldsymbol{\eta}_u^{n-1}\|_0^2 - \|2\delta_t \boldsymbol{\eta}_u^{n-1} - \delta_t \boldsymbol{\eta}_u^{n-2}\|_0^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 4\Delta t\nu\|\nabla\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + 4\Delta t\gamma\|\nabla\cdot\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + \frac{4(\Delta t)^2}{3}\|\nabla\delta_t\eta_p^n\|_0^2 \\
\le\ & 4\Delta t(\delta_t\boldsymbol{R}^n, \delta_t\widetilde{\boldsymbol{\eta}}_u^n) + \frac{8}{3}(\Delta t)^2(\nabla\delta_t(p(t_n) - p(t_{n-1})), \nabla\delta_t\eta_p^{n-1}) \\
&+ \frac{4}{3}(\Delta t)^2\|\nabla\delta_t\eta_p^{n-1}\|_0^2 + 3\|\delta_t\boldsymbol{\eta}_u^n - \delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 \\
&+ \frac{8}{3}(\Delta t)^2(\nabla\delta_{tt}p(t_n), \nabla\delta_t(\eta_p^n - \eta_p^{n-1}))
\end{aligned}
$$

and using $\|\boldsymbol{\eta}_u^n\|_0 \overset{1.2.1}{\le} \|\widetilde{\boldsymbol{\eta}}_u^n\|_0$ we derive

$$
\begin{aligned}
&\|\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + \|2\delta_t\boldsymbol{\eta}_u^n - \delta_t\boldsymbol{\eta}_u^{n-1}\|_0^2 \\
&\quad + \|\delta_{ttt}\boldsymbol{\eta}_u^n\|_0^2 - \|\delta_t\widetilde{\boldsymbol{\eta}}_u^{n-1}\|_0^2 - \|2\delta_t\boldsymbol{\eta}_u^{n-1} - \delta_t\boldsymbol{\eta}_u^{n-2}\|_0^2 \\
&\quad + 4\Delta t\nu\|\nabla\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + 4\Delta t\gamma\|\nabla\cdot\delta_t\widetilde{\boldsymbol{\eta}}^n\|_0^2 + \frac{4}{3}(\Delta t)^2\|\nabla\delta_t\eta_p^n\|_0^2 \\
&\le \frac{4}{3}(\Delta t)^2\|\nabla\delta_t\eta_p^{n-1}\|_0^2 + 4\Delta t(\delta_t\boldsymbol{R}^n, \delta_t\widetilde{\boldsymbol{\eta}}_u^n) + 4\Delta t(\nabla\delta_{tt}p(t_n), \delta_t\widetilde{\boldsymbol{\eta}}_u^n) \\
&\quad + \frac{8}{3}(\Delta t)^2(\nabla\delta_{tt}p(t_n), \nabla\delta_t\eta_p^n) \\
&\le \frac{4}{3}(\Delta t)^2\|\nabla\delta_t\eta_p^{n-1}\|_0^2 + 4\Delta t\|\delta_t\boldsymbol{R}^n\|_0^2 + \Delta t\|\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 \\
&\quad + 4\Delta t\|\nabla\delta_{tt}p(t_n)\|_0^2 + \Delta t\|\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + \frac{8}{3}\Delta t\|\nabla\delta_{tt}p(t_n)\|_0^2 \\
&\quad + \frac{2}{3}(\Delta t)^3\|\nabla\delta_t\eta_p^n\|_0^2.
\end{aligned}
$$

Next, we sum this equation up from $n = 3$ to $m \le N$:

$$
\begin{aligned}
&\|\delta_t\widetilde{\boldsymbol{\eta}}_u^m\|_0^2 + \|2\delta_t\boldsymbol{\eta}_u^m - \delta_t\boldsymbol{\eta}_u^{m-1}\|_0^2 + \frac{4}{3}(\Delta t)^2\|\nabla\delta_t\eta_p^m\|_0^2 \\
&\quad + \sum_{n=3}^{m}\left(\|\delta_{ttt}\boldsymbol{\eta}_u^n\|_0^2 + 4\Delta t\nu\|\nabla\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + 4\Delta t\gamma\|\nabla\cdot\delta_t\widetilde{\boldsymbol{\eta}}^n\|_0^2\right) \\
&\le \|\delta_t\widetilde{\boldsymbol{\eta}}_u^2\|_0^2 + \|2\delta_t\boldsymbol{\eta}_u^2 - \delta_t\boldsymbol{\eta}_u^1\|_0^2 + \frac{4}{3}(\Delta t)^2\|\nabla\delta_t\eta_p^2\|_0^2 \\
&\quad + \Delta t\sum_{n=3}^{m}\left(2\|\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + \frac{2}{3}(\Delta t)^2\|\nabla\delta_t\eta_p^n\|_0^2\right) \\
&\quad + \Delta t\sum_{n=3}^{m}\left(4\|\delta_t\boldsymbol{R}^n\|_0^2 + 7\|\partial_t^2 p(t_n)\|_1^2(\Delta t)^4\right).
\end{aligned}
$$

Provided that $2\Delta t \le 1$, the discrete Gronwall lemma for $\|\delta_t\widetilde{\boldsymbol{\eta}}_u^m\|_0^2 +$

$\frac{4}{3}(\Delta t)^2\|\nabla\delta_t\eta_p^m\|_0^2$ yields in combination with the initial estimates

$$\begin{aligned}
&\|\delta_t\widetilde{\boldsymbol{\eta}}_u^m\|_0^2 + \|2\delta_t\boldsymbol{\eta}_u^m - \delta_t\boldsymbol{\eta}_u^{m-1}\|_0^2 + \frac{4}{3}(\Delta t)^2\|\nabla\delta_t\eta_p^m\|_0^2 \\
&\quad + \sum_{n=3}^{m}\left(\|\delta_{ttt}\boldsymbol{\eta}_u^n\|_0^2 + 4\Delta t\nu\|\nabla\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + 4\Delta t\gamma\|\nabla\cdot\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2\right) \\
&\leq \|\delta_t\widetilde{\boldsymbol{\eta}}_u^2\|_0^2 + \|2\delta_t\boldsymbol{\eta}_u^2 - \delta_t\boldsymbol{\eta}_u^1\|_0^2 + \frac{4}{3}(\Delta t)^2\|\nabla\delta_t\eta_p^2\|_0^2 \\
&\quad + C\Delta t\sum_{n=3}^{m}\left(\|\delta_t\boldsymbol{R}^n\|_0^2 + (\Delta t)^4\right) \leq C(\Delta t)^4,
\end{aligned} \tag{2.2.14}$$

since $\|\delta_t\boldsymbol{R}^n\|_0^2 \leq C(\Delta t)^4$.

The projection error equation states

$$\begin{aligned}
\|\boldsymbol{\eta}_u^m - \widetilde{\boldsymbol{\eta}}_u^m\|_0 &= \frac{2\Delta t}{3}\|\nabla(\delta_t\eta_p^n - \delta_t p(t_{k+1}))\|_0 \\
&\leq \frac{2\Delta t}{3}(\|\nabla\delta_t\eta_p^n\|_0 + \|\delta_t p(t_{k+1})\|_0) \\
&\leq C(\Delta t)^2.
\end{aligned}$$

□

Corollary 2.2.4. *For all* $1 \leq m \leq N$ *it holds*

$$\nu\|\nabla\widetilde{\boldsymbol{\eta}}_u^m\|_0^2 + \gamma\|\nabla\cdot\widetilde{\boldsymbol{\eta}}_u^m\|_0^2 \leq C\Delta t^2. \tag{2.2.15}$$

Proof. Due to (2.2.14) we have

$$\sum_{n=3}^{m}(\nu\|\nabla\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + \gamma\|\nabla\cdot\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2) \leq C(\Delta t)^3$$

and therefore via triangle inequality and initial error estimates

$$\begin{aligned}
&\sqrt{\nu}\|\nabla\widetilde{\boldsymbol{\eta}}_u^m\|_0 + \sqrt{\gamma}\|\nabla\cdot\widetilde{\boldsymbol{\eta}}_u^m\|_0 \leq \sum_{n=1}^{m}(\sqrt{\nu}\|\nabla\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0 + \sqrt{\gamma}\|\nabla\cdot\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0) \\
&\leq C\left(m\sum_{n=1}^{m}(\nu\|\nabla\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + \gamma\|\nabla\cdot\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2)\right)^{1/2} \leq C\Delta t.
\end{aligned}$$

due to $m \leq N = 1/(\Delta t)$. □

Lemma 2.2.5. *For all* $1 \leq m \leq N$ *it holds*

$$\|\widetilde{\boldsymbol{\eta}}_u^m\|_0^2 \leq c(\Delta t)^4. \tag{2.2.16}$$

Proof. Next we want to bound $\|\widetilde{\boldsymbol{\eta}}_u^n\|_0$. Therefore we eliminate $\boldsymbol{\eta}_u^n$ in the error equation corresponding to (2.2.2) to get

$$\begin{aligned}
&\left(\frac{3\widetilde{\boldsymbol{\eta}}_u^n - 4\widetilde{\boldsymbol{\eta}}_u^{n-1} + \widetilde{\boldsymbol{\eta}}_u^{n-2}}{2\Delta t}, \boldsymbol{v}\right) + \nu(\nabla\widetilde{\boldsymbol{\eta}}_u^n, \nabla\boldsymbol{v}) + \gamma(\nabla\cdot\widetilde{\boldsymbol{\eta}}_u^n, \nabla\cdot\boldsymbol{v}) - (\boldsymbol{R}^n, \boldsymbol{v})\\
&= (\nabla(-p(t_n) + \frac{7}{3}r^{n-1} - \frac{5}{3}r^{n-2} + \frac{1}{3}r^{n-3}), \boldsymbol{v})\\
&=: (\nabla\zeta^n, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}.
\end{aligned}$$

Now we test the equation with the inverse Stokes operator (A.4.1) applied to $4\Delta t\widetilde{\boldsymbol{\eta}}_u^n$:

$$\begin{aligned}
&(2(3\widetilde{\boldsymbol{\eta}}_u^n - 4\widetilde{\boldsymbol{\eta}}_u^{n-1} + \widetilde{\boldsymbol{\eta}}_u^{n-2}), S\widetilde{\boldsymbol{\eta}}_u^n) + 4\Delta t\nu(\nabla\widetilde{\boldsymbol{\eta}}_u^n, \nabla S\widetilde{\boldsymbol{\eta}}_u^n) + 4\Delta t\gamma(\nabla\cdot\widetilde{\boldsymbol{\eta}}_u^n, \nabla\cdot S\widetilde{\boldsymbol{\eta}}_u^n)\\
&= 4\Delta t(\boldsymbol{R}^n, S\widetilde{\boldsymbol{\eta}}_u^n) + 4\Delta t(\nabla\zeta^n, S\widetilde{\boldsymbol{\eta}}_u^n) = 4\Delta t(\boldsymbol{R}^n, S\widetilde{\boldsymbol{\eta}}_u^n).
\end{aligned}$$

using $S\widetilde{\boldsymbol{\eta}}_u^n \in \boldsymbol{V}^{div}$. Recall that S is a self-adjoint operator:

$$\begin{aligned}
(\boldsymbol{v}, S\boldsymbol{w}) &= \nu(\nabla S\boldsymbol{v}, \nabla S\boldsymbol{w}) + \gamma(\nabla\cdot S\boldsymbol{v}, \nabla\cdot S\boldsymbol{w}) + (r_v, S\boldsymbol{w})\\
&= \nu(\nabla S\boldsymbol{w}, \nabla S\boldsymbol{v}) + \gamma(\nabla\cdot S\boldsymbol{w}, \nabla\cdot S\boldsymbol{v}) + (r_w, S\boldsymbol{w})\\
&= (\boldsymbol{v}, S\boldsymbol{w}).
\end{aligned}$$

By the definition of the induced semi-norm $|\boldsymbol{u}|_* = (\boldsymbol{u}, S\boldsymbol{u})$ and the splitting (A.1.1) we get

$$\begin{aligned}
&|\widetilde{\boldsymbol{\eta}}_u^n|_*^2 + |2\widetilde{\boldsymbol{\eta}}_u^n - \widetilde{\boldsymbol{\eta}}_u^{n-1}|_*^2 + |\delta_{tt}\widetilde{\boldsymbol{\eta}}_u^n|_*^2\\
&\quad + 4\Delta t\nu(\nabla\widetilde{\boldsymbol{\eta}}_u^n, \nabla S\widetilde{\boldsymbol{\eta}}_u^n) + 4\Delta t\gamma(\nabla\cdot\widetilde{\boldsymbol{\eta}}_u^n, \nabla\cdot S\widetilde{\boldsymbol{\eta}}_u^n)\\
&= 4\Delta t(\boldsymbol{R}^n, S\widetilde{\boldsymbol{\eta}}_u^n) + |\widetilde{\boldsymbol{\eta}}_u^{n-1}|_*^2 + |2\widetilde{\boldsymbol{\eta}}_u^{n-1} - \widetilde{\boldsymbol{\eta}}_u^{n-2}|_*^2.
\end{aligned} \tag{2.2.17}$$

The consistency error can be bounded as

$$\begin{aligned}
4\Delta t(\boldsymbol{R}^n, S\widetilde{\boldsymbol{\eta}}_u^n) &\leq 4\frac{\Delta t}{\nu}\|\boldsymbol{R}^n\|_{-1}^2 + \Delta t\nu\|S\widetilde{\boldsymbol{\eta}}_u^n\|_1^2\\
&\overset{(A.4.5)}{\leq} 4\frac{\Delta t}{\nu}\|\boldsymbol{R}^n\|_{-1}^2 + \Delta t\|\widetilde{\boldsymbol{\eta}}_u^n\|_{-1}^2\\
&\leq 4\frac{\Delta t}{\nu}\|\boldsymbol{R}^n\|_{-1}^2 + \Delta t\|\widetilde{\boldsymbol{\eta}}_u^n\|_0^2.
\end{aligned}$$

Using (A.4.7) with $\epsilon = 2\left(\frac{2\nu+\gamma}{\nu}\right)^{-2}$, the diffusive term and the grad-div stabilization can be estimated by

$$4\Delta t\nu(\nabla\widetilde{\boldsymbol{\eta}}_u^n, \nabla S\widetilde{\boldsymbol{\eta}}_u^n) + 4\Delta t\gamma(\nabla\cdot\widetilde{\boldsymbol{\eta}}_u^n, \nabla\cdot S\widetilde{\boldsymbol{\eta}}_u^n) \geq 2\Delta t\|\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 - c\Delta t\|\widetilde{\boldsymbol{\eta}}_u^n - \boldsymbol{\eta}_u^n\|_0^2.$$

where $c = 2\left(\frac{2\nu+\gamma}{\nu}\right)^2$. Combining these estimates we arrive at

$$|\widetilde{\boldsymbol{\eta}}_u^n|_*^2 + |2\widetilde{\boldsymbol{\eta}}_u^n - \widetilde{\boldsymbol{\eta}}_u^{n-1}|_*^2 + |\delta_{tt}\widetilde{\boldsymbol{\eta}}_u^n|_*^2 + \Delta t\|\widetilde{\boldsymbol{\eta}}_u^n\|_0^2$$

$$\leq 4\frac{\Delta t}{\nu}\|\boldsymbol{R}^n\|_{-1}^2 + |\widetilde{\boldsymbol{\eta}}_u^{n-1}|_*^2 + |2\widetilde{\boldsymbol{\eta}}_u^{n-1} - \widetilde{\boldsymbol{\eta}}_u^{n-2}|_*^2 + c\Delta t\|\widetilde{\boldsymbol{\eta}}_u^n - \boldsymbol{\eta}_u^n\|_0^2$$

that yields summed up

$$\begin{aligned}
&|\widetilde{\boldsymbol{\eta}}_u^m|_*^2 + |2\widetilde{\boldsymbol{\eta}}_u^m - \widetilde{\boldsymbol{\eta}}_u^{m-1}|_*^2 + \sum_{n=3}^{m}(|\delta_{tt}\widetilde{\boldsymbol{\eta}}_u^n|_*^2 + \Delta t\|\widetilde{\boldsymbol{\eta}}_u^n\|_0^2) \\
&\leq |\widetilde{\boldsymbol{\eta}}_u^2|_*^2 + |2\widetilde{\boldsymbol{\eta}}_u^2 - \widetilde{\boldsymbol{\eta}}_u^1|_*^2 + \sum_{n=3}^{m}\left(4\frac{\Delta t}{\nu}\|\boldsymbol{R}^n\|_{-1}^2 + c\Delta t\|\widetilde{\boldsymbol{\eta}}_u^n - \boldsymbol{\eta}_u^n\|_0^2\right) \\
&\leq c(\Delta t)^4.
\end{aligned} \tag{2.2.18}$$

In particular, we derive

$$\Delta t\sum_{n=1}^{m}\|\widetilde{\boldsymbol{\eta}}_u\|_{l^2(t_0,T;[L^2(\Omega)]^d)}^2 = \Delta t\sum_{n=0}^{N}\|\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 \leq c(\Delta t)^4. \tag{2.2.19}$$

In order to get an l^∞-estimate we use (A.4.7) again with $\epsilon = 2\left(\frac{\nu}{2\nu+\gamma}\right)^2$ and obtain from Lemma (2.2.3) together with (2.2.18)

$$\begin{aligned}
\|\widetilde{\boldsymbol{\eta}}_u\|_{l^\infty(t_0,T;[L^2(\Omega)]^d)}^2 &\leq C\max_{1\leq n\leq N}|\widetilde{\boldsymbol{\eta}}_u^n|_*^2 + C\left(1+\frac{\gamma}{\nu}\right)^2\max_{1\leq n\leq N}\|\widetilde{\boldsymbol{\eta}}_u^n - \boldsymbol{\eta}_u^n\|_0^2 \\
&\leq \left(\frac{C}{\nu} + \left(1+\frac{\gamma}{\nu}\right)^2 C_{G,\text{lin}}\right)(\Delta t)^4.
\end{aligned}$$

□

2.3 Spatial Discretization

Now that we considered the discretization time, we finalize this approach with the discretization in time. The problem that we consider in this section reads:

Find $\widetilde{\boldsymbol{u}}_{ht}^n \in \boldsymbol{V}_h, \boldsymbol{u}_{ht}^n \in \boldsymbol{Y}_h$ and $p_{ht}^n \in Q_h$ such that

$$\begin{aligned}
&\left(\frac{3\widetilde{\boldsymbol{u}}_{ht}^n - 4\boldsymbol{u}_{ht}^{n-1} + \boldsymbol{u}_{ht}^{n-2}}{2\Delta t}, \boldsymbol{v}_h\right) + \nu(\nabla\widetilde{\boldsymbol{u}}_{ht}^n, \nabla\boldsymbol{v}_h) \\
&\quad + c(\widetilde{\boldsymbol{u}}_{ht}^n; \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) + s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) \\
&\quad + \gamma(\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^n, \nabla\cdot\boldsymbol{v}_h) = (\boldsymbol{f}^n, \boldsymbol{v}_h) - (\nabla p_{ht}^{n-1}, \boldsymbol{v}_h) \\
&\widetilde{\boldsymbol{u}}_{ht}^n|_{\partial\Omega} = 0,
\end{aligned} \tag{2.3.1}$$

$$\begin{aligned}
\left(\frac{3\boldsymbol{u}_{ht}^n - 3\widetilde{\boldsymbol{u}}_{ht}^n}{2\Delta t} + \nabla(p_{ht}^n - p_{ht}^{n-1}), \boldsymbol{y}_h\right) &= 0 \\
(\nabla\cdot\boldsymbol{u}_{ht}^n, q_h) &= 0, \\
\boldsymbol{u}_{ht}^n|_{\partial\Omega} &= 0
\end{aligned} \tag{2.3.2}$$

holds for all $\boldsymbol{v}_h \in \boldsymbol{V}_h, \boldsymbol{y}_h \in \boldsymbol{Y}_h$ and $q_h \in Q_h$.

We compare this with the discretization in time:

Find $\widetilde{\boldsymbol{w}}_t^n \in \boldsymbol{V}, \boldsymbol{w}_t^n \in \boldsymbol{V}^{div}$ and $r_t^n \in Q$ such that

$$\begin{aligned}
&\left(\frac{3\widetilde{\boldsymbol{w}}_t^n - 4\boldsymbol{w}_t^{n-1} + \boldsymbol{w}_t^{n-2}}{2\Delta t}, \boldsymbol{v}\right) + \nu(\nabla \widetilde{\boldsymbol{w}}_t^n, \nabla \boldsymbol{v}) + \gamma(\nabla \cdot \widetilde{\boldsymbol{u}}_t^n, \nabla \cdot \boldsymbol{v}) \\
&\quad = (\boldsymbol{f}^n, \boldsymbol{v}) - (\nabla r_t^{n-1}, \boldsymbol{v}) - c(\boldsymbol{u}(t_n); \boldsymbol{u}(t_n), \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V} \\
&\widetilde{\boldsymbol{w}}_t^n|_{\partial\Omega} = 0,
\end{aligned} \tag{2.3.3}$$

$$\begin{aligned}
\left(\frac{3\boldsymbol{w}_t^n - 3\widetilde{\boldsymbol{w}}_t^n}{2\Delta t} + \nabla(r_t^n - r_t^{n-1}), \boldsymbol{y}\right) &= 0 \quad \forall \boldsymbol{y} \in [L^2(\Omega)]^d, \\
(\nabla \cdot \boldsymbol{w}_t^n, q) &= 0 \quad \forall q \in Q, \\
\boldsymbol{w}_t^n|_{\partial\Omega} &= 0.
\end{aligned} \tag{2.3.4}$$

2.3.1 Notation

We use the abbreviations

$$\widetilde{\boldsymbol{\eta}}_{u,h}^n := \widetilde{\boldsymbol{w}}_t^n - j_u \widetilde{\boldsymbol{w}}_t^n, \quad \boldsymbol{\eta}_{u,h}^n := \boldsymbol{w}_t^n - j_u \boldsymbol{w}_t^n, \quad \eta_{p,h}^n := r_t^n - j_p r_t^n$$

for the approximation errors and

$$\widetilde{\boldsymbol{e}}_{u,h}^n := j_u \widetilde{\boldsymbol{w}}_t^n, \quad \boldsymbol{e}_{u,h}^n := j_u \boldsymbol{w}_t^n - \boldsymbol{u}_{ht}^n, \quad e_{p,h}^n := j_p r_t^n - p_{ht}^n$$

for the discretization errors. That is the errors due to spatial discretization can be written as

$$\begin{aligned}
&\widetilde{\boldsymbol{\xi}}_{u,h}^n := \widetilde{\boldsymbol{w}}_t^n - \widetilde{\boldsymbol{u}}_{ht}^n = \widetilde{\boldsymbol{\eta}}_{u,h}^n + \widetilde{\boldsymbol{e}}_{u,h}^n, \quad \boldsymbol{\xi}_{u,h}^n := \boldsymbol{w}_t^n - \boldsymbol{u}_{ht}^n = \boldsymbol{\eta}_{u,h}^n + \boldsymbol{e}_{u,h}^n, \\
&\xi_{p,h}^n := r_t^n - p_{ht}^n = \eta_{p,h}^n + e_{p,h}^n.
\end{aligned}$$

2.3.2 The Interpolation Operators

We choose $j_u \boldsymbol{u}_t^n := \boldsymbol{w}_{ht}^n$ as solution of the Stokes problem

Find $\boldsymbol{w}_{ht}^n \in \boldsymbol{V}_h$ and $\widetilde{r}_{ht}^{n-1} \in Q_h$ such that

$$\begin{aligned}
&\nu(\nabla \boldsymbol{w}_{ht}^n, \nabla \boldsymbol{v}_h) + \gamma(\nabla \cdot \boldsymbol{w}_{ht}^n, \nabla \cdot \boldsymbol{v}_h) - (\widetilde{r}_{ht}^{n-1}, \nabla \cdot \boldsymbol{v}_h) \\
&= \nu(\nabla \boldsymbol{w}_t^n, \nabla \boldsymbol{v}_h) + \gamma(\nabla \cdot \boldsymbol{w}_t^n, \nabla \cdot \boldsymbol{v}_h) - (r_t^{n-1}, \nabla \cdot \boldsymbol{v}_h) \\
&(\nabla \cdot \boldsymbol{w}_{ht}^n, q_h) = (\nabla \cdot \boldsymbol{w}_t^n, q_h)
\end{aligned} \tag{2.3.5}$$

holds for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $q_h \in Q_h$.

and $(j_u \widetilde{\boldsymbol{u}}_t^n := \widetilde{\boldsymbol{w}}_{ht}, j_p p_t = r_{ht})$ is given as solution of the Stokes problem

Find $\widetilde{\boldsymbol{w}}_{ht}^n \in \boldsymbol{V}_h$ and $r_{ht}^{n-1} \in Q_h$ such that

$$\begin{aligned}
&\nu(\nabla \widetilde{\boldsymbol{w}}_{ht}^n, \nabla \boldsymbol{v}_h) + \gamma(\nabla \cdot \widetilde{\boldsymbol{w}}_{ht}^n, \nabla \cdot \boldsymbol{v}_h) - (r_{ht}^{n-1}, \nabla \cdot \boldsymbol{v}_h) \\
&= \nu(\nabla \widetilde{\boldsymbol{w}}_t^n, \nabla \boldsymbol{v}_h) + \gamma(\nabla \cdot \widetilde{\boldsymbol{w}}_t^n, \nabla \cdot \boldsymbol{v}_h) - (r_t^{n-1}, \nabla \cdot \boldsymbol{v}_h) \\
&(\nabla \cdot \boldsymbol{w}_{ht}^n, q_h) = (\nabla \cdot \widetilde{\boldsymbol{w}}_t^n, q_h)
\end{aligned} \tag{2.3.6}$$

holds for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $q_h \in Q_h$.

According to [11, Theorem 1] the solution of the grad-div stabilized Stokes problem can be bounded by

$$\begin{aligned}
\nu\|\nabla \widetilde{\boldsymbol{\eta}}_u^n\|_0^2 &\lesssim \inf_{\boldsymbol{w}_h \in \boldsymbol{V}_h^{div}} \left(\nu\|\nabla(\widetilde{\boldsymbol{u}}_t^n - \boldsymbol{w}_h)\|_0^2 + \gamma\|\nabla \cdot (\widetilde{\boldsymbol{u}}_t^n - \boldsymbol{w}_h)\|_0^2\right) \\
&\quad + \gamma^{-1} \inf_{q_h \in Q_h} \|p_t^n - q_h\|_0^2 \\
&\lesssim \inf_{\boldsymbol{w}_h \in \boldsymbol{V}_h^{div}} (\nu + \gamma)\|\nabla(\widetilde{\boldsymbol{u}}_t^n - \boldsymbol{w}_h)\|_0^2 + \frac{1}{\gamma} \inf_{q_h \in Q_h} \|p_t^n - q_h\|_0^2.
\end{aligned}$$

This result can easily be extended to include the grad-div stabilization on the left-hand side and using inf-sup stability we arrive at

$$\begin{aligned}
&\nu\|\nabla \widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + \gamma\|\nabla \cdot \widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + \|e_p^n\|_0^2 \\
&\leq C\left(\inf_{\boldsymbol{w}_h \in \boldsymbol{V}_h^{div}} (\nu + \gamma)\|\nabla(\widetilde{\boldsymbol{u}}_t^n - \boldsymbol{w}_h)\|_0^2 + \frac{1}{\gamma} \inf_{q_h \in Q_h} \|p_t^n - q_h\|_0^2\right).
\end{aligned}$$

Using interpolation results according to Assumption 1.1.3 and the local inverse inequality (Assumption 1.1.2) we get the bound

$$\begin{aligned}
&\nu\|\widetilde{\boldsymbol{\eta}}_u^n\|_1^2 + \gamma\|\nabla \cdot \widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + \|\eta_p^n\|_0^2 + h^2\|\nabla \eta_p^n\|_0^2 \\
&\leq C((\nu + \gamma)h^{2k_u}\|\widetilde{\boldsymbol{u}}_t^n\|_{W^{k_u+1,2}}^2 + \gamma^{-1}h^{2k_p+2}\|p_t^n\|_{W^{k_p+1,2}}^2)
\end{aligned} \tag{2.3.7}$$

and using the Aubin-Nitsche trick

$$\begin{aligned}
\|\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 &\leq Ch^2(\nu\|\widetilde{\boldsymbol{\eta}}_u^n\|_1^2 + \gamma\|\nabla \cdot \widetilde{\boldsymbol{\eta}}_u^n\|_0^2) \\
&\leq C((\nu + \gamma)h^{2k_u+2}\|\widetilde{\boldsymbol{u}}_t^n\|_{W^{k_u+1,2}}^2 + \frac{1}{\gamma}h^{2k_p+4}\|p_t^n\|_{W^{k_p+1,2}}^2).
\end{aligned} \tag{2.3.8}$$

This gives us stability according to

$$\begin{aligned}
&\max_{1\leq n\leq N}(\|\boldsymbol{w}_{ht}^n\|_0^2 + \nu\|\boldsymbol{w}_{ht}^n\|_1^2 + \gamma\|\nabla \cdot \boldsymbol{w}_{ht}^n\|_0^2 + \|p_{ht}^n\|_0^2) \\
&\leq \max_{1\leq n\leq N}(\|\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + \nu\|\nabla \widetilde{\boldsymbol{\eta}}_u^n\|_1^2 + \gamma\|\nabla \cdot \widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + \|\eta_p^n\|_0^2 \\
&\qquad + \|\widetilde{\boldsymbol{u}}_t^n\|_0^2 + \nu\|\nabla \widetilde{\boldsymbol{u}}_t^n\|_1^2 + \gamma\|\nabla \cdot \widetilde{\boldsymbol{u}}_t^n\|_0^2 + \|p_t^n\|_0^2) \\
&\leq C \max_{1\leq n\leq N}((\|\widetilde{\boldsymbol{u}}_t^n\|_0^2 + \nu\|\nabla \widetilde{\boldsymbol{u}}_t^n\|_1^2 + \gamma\|\nabla \cdot \widetilde{\boldsymbol{u}}_t^n\|_0^2 + \|p_t^n\|_0^2)) \leq C.
\end{aligned} \tag{2.3.9}$$

For $\boldsymbol{\eta}_u^n$ we achieve analogous estimates.

If the time discretized solutions are not sufficiently smooth, we use instead of the approximation properties of the Stokes interpolant the splitting $\widetilde{\boldsymbol{u}}_t^n = \boldsymbol{u}(t_n) - \widetilde{\boldsymbol{\xi}}_u^n$. Combined with Assumption 1.1.3 and the time convergence results of Section 2.2 we obtain

$$\begin{aligned}
\|\widetilde{\boldsymbol{\eta}}_{u,h}^1\|_0^2 &\le \|\widetilde{\boldsymbol{\eta}}_{u,h}^1\|_0^2 = \|(\boldsymbol{u}(t_1) - \widetilde{\boldsymbol{\eta}}_u^1) - j_{u,h}(\boldsymbol{u}(t_1) - \widetilde{\boldsymbol{\eta}}_u^1)\|_0^2 \\
&\le C\|\boldsymbol{u}(t_1) - j_{u,h}\boldsymbol{u}(t_1)\|_0^2 + C\|\widetilde{\boldsymbol{\eta}}_u^1 - j_{u,h}\widetilde{\boldsymbol{\eta}}_u^1\|_0^2 \\
&\le C\|\boldsymbol{u}(t_1) - j_{u,h}\boldsymbol{u}(t_1)\|_0^2 + Ch^2\|\nabla\widetilde{\boldsymbol{\eta}}_u^1\|_1^2 \\
&\le C(\nu+\gamma)h^{2k_u+2} + C\gamma^{-1}h^{2k_p+4} + C\Delta t^2 h^2\left(\frac{\nu+\gamma}{\nu} + \frac{1}{\nu\gamma}\right), \\
\nu\|\nabla\widetilde{\boldsymbol{\eta}}_{u,h}^1\|_0^2 &\le C\nu\|\nabla(\boldsymbol{u}(t_1) - j_{u,h}\boldsymbol{u}(t_1))\|_0^2 + C\nu\|\nabla(\widetilde{\boldsymbol{\eta}}_u^1 - j_{u,h}\widetilde{\boldsymbol{\eta}}_u^1)\|_0^2 \\
&\le C\nu\|\nabla(\boldsymbol{u}(t_1) - j_{u,h}\boldsymbol{u}(t_1))\|_0^2 + C\nu\|\nabla\widetilde{\boldsymbol{\eta}}_u^1\|_0^2 \\
&\le C(\nu+\gamma)h^{2k_u} + C\gamma^{-1}h^{2k_p+2} + C\Delta t^2(\nu+\gamma+\gamma^{-1})
\end{aligned}$$

and analogously for $\boldsymbol{\eta}_{u,h}^n$ and $\eta_{p,h}^n$.

2.3.3 Initial Errors

For the first time step we use a BDF1 instead of the BDF2 scheme. In particular, the convection-diffusion step and the projection in the fully discretized setting read

Find $\widetilde{\boldsymbol{u}}_{ht}^1 \in \boldsymbol{V}_h, \boldsymbol{u}_{ht}^1 \in \boldsymbol{Y}_h$ and $p_{ht}^1 \in Q_h$ such that

$$\begin{aligned}
&\left(\frac{\widetilde{\boldsymbol{u}}_{ht}^1 - \boldsymbol{u}_{ht}^0}{\Delta t}, \boldsymbol{v}_h\right) + \nu(\nabla\widetilde{\boldsymbol{u}}_{ht}^1, \nabla\boldsymbol{v}_h) + c(\widetilde{\boldsymbol{u}}_{ht}^1; \widetilde{\boldsymbol{u}}_{ht}^1, \boldsymbol{v}_h) \\
&\quad + s_h(\widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{u}}_{ht}^1, \boldsymbol{v}_h) + \gamma(\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^1, \nabla\cdot\boldsymbol{v}_h) \\
&\quad = (\boldsymbol{f}^1, \boldsymbol{v}_h) - (\nabla p_{ht}^0, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h \\
&\widetilde{\boldsymbol{u}}_{ht}^1|_{\partial\Omega} = 0,
\end{aligned} \tag{2.3.10}$$

$$\begin{aligned}
\left(\frac{\boldsymbol{u}_{ht}^1 - \widetilde{\boldsymbol{u}}_{ht}^1}{\Delta t} + \nabla(p_{ht}^1 - p_{ht}^0), \nabla\boldsymbol{y}_h\right) &= 0 \quad \forall \boldsymbol{y}_{\boldsymbol{h}} \in \boldsymbol{Y}_h \\
(\nabla\cdot\boldsymbol{u}_{ht}^1, q_h) &= 0 \quad \forall q_h \in Q_h, \\
\boldsymbol{u}_{ht}^1|_{\partial\Omega} &= 0
\end{aligned} \tag{2.3.11}$$

with the initial values $\widetilde{\boldsymbol{u}}_{ht}^0 = j_u\widetilde{\boldsymbol{w}}_t^0, \boldsymbol{u}_{ht}^0 = j_u\boldsymbol{w}_t^0$ and $p_{ht}^0 = j_p r_t^0$.

Therefore the initial discretization errors vanish: $\|\widetilde{\boldsymbol{e}}_{u,h}^0\| = \|\boldsymbol{e}_{u,h}^0\| = \|\eta_p^0\| = 0$.

The technique used in the next Lemma is the same as later for the case $n \ge 2$. For the sake of completeness, we nevertheless write it down at both places.

Lemma 2.3.1. *The initial errors due to spatial discretization are bounded by*

$$
\begin{aligned}
&C\,(1-K)\,\|\widetilde{\boldsymbol{e}}^1_{u,h}\|_0^2 + \nu\Delta t\|\nabla\widetilde{\boldsymbol{e}}^1_{u,h}\|_0^2 \\
&\quad + \gamma\Delta t\|\nabla\cdot\widetilde{\boldsymbol{e}}^1_{u,h}\|_0^2 + 2\Delta t\sum_{M\in\mathcal{M}_h}\tau_M^1\|\kappa_M((\widetilde{\boldsymbol{u}}^1_{ht}\cdot\nabla)\widetilde{\boldsymbol{e}}^1_{u,h})\|_{0,M}^2 \\
&\lesssim \left((\nu+\gamma)h^{2k_u+2} + \gamma^{-1}h^{2k_p+4} + \Delta t^2h^2\left(\frac{\nu+\gamma}{\nu}+\frac{1}{\nu\gamma}\right)\right)\left(1+\frac{\Delta t}{\nu}\right) \\
&\quad + \Delta t\frac{(\nu+\gamma)^2h^{4k_u} + \gamma^{-2}h^{4k_p+4} + \Delta t^4(\nu+\gamma+\gamma^{-1})^2}{\nu^3} \\
&\quad + \Delta t\max_{M\in\mathcal{M}_h}\{\tau_M^1|\widetilde{\boldsymbol{u}}^1_M|^2\}\left(\frac{\nu+\gamma}{\nu}h^{2k_u} + \frac{h^{2k_p+2}}{\nu\gamma} + \Delta t^2\frac{\nu+\gamma+\gamma^{-1}}{\nu} + h^{2s}\right)
\end{aligned}
\tag{2.3.12}
$$

where K is defined according to

$$
K := \frac{\Delta t}{\nu} - \Delta t\frac{(\nu+\gamma)^2h^{4k_u} + \gamma^{-2}h^{4k_p+4} + \Delta t^4(\nu+\gamma+\gamma^{-1})^2}{\nu^5}.
$$

Proof. Testing the difference of the convection-diffusion equations with $\widetilde{\boldsymbol{e}}^1_{u,h}$ gives

$$
\begin{aligned}
&\|\widetilde{\boldsymbol{e}}^1_{u,h}\|_0^2 + \nu\Delta t\|\nabla\widetilde{\boldsymbol{e}}^1_{u,h}\|_0^2 + \gamma\Delta t\|\nabla\cdot\widetilde{\boldsymbol{e}}^1_{u,h}\|_0^2 \\
&\quad + \Delta t\sum_{M\in\mathcal{M}_h}\left(\tau_M^1\|\kappa_M((\widetilde{\boldsymbol{u}}^1_{ht}\cdot\nabla)\widetilde{\boldsymbol{e}}^1_{u,h})\|_0^2\right) \\
&= \Delta t(c(\widetilde{\boldsymbol{u}}^1_{ht},\widetilde{\boldsymbol{u}}^1_{ht},\widetilde{\boldsymbol{e}}^1_{u,h}) - c(\widetilde{\boldsymbol{w}}^1_t,\widetilde{\boldsymbol{w}}^1_t,\widetilde{\boldsymbol{e}}^1_{u,h})) \\
&\quad + \Delta t(s_h(\widetilde{\boldsymbol{u}}^1_{ht},\widetilde{\boldsymbol{u}}^1_{ht},\widetilde{\boldsymbol{u}}^1_{ht},\widetilde{\boldsymbol{e}}^1_{u,h}) + s_h(\widetilde{\boldsymbol{u}}^1_{ht},\widetilde{\boldsymbol{e}}^1_{u,h},\widetilde{\boldsymbol{u}}^1_{ht},\widetilde{\boldsymbol{e}}^1_{u,h})) \\
&\quad - \Delta t(\nabla\eta^0_{p,h},\widetilde{\boldsymbol{e}}^1_{u,h}) - (\widetilde{\boldsymbol{\eta}}^1_{u,h} - \boldsymbol{\eta}^0_{u,h},\widetilde{\boldsymbol{e}}^1_{u,h}) \\
&\quad - \nu\Delta t(\nabla\widetilde{\boldsymbol{\eta}}^1_{u,h},\nabla\widetilde{\boldsymbol{e}}^1_{u,h}) - \gamma\Delta t(\nabla\cdot\widetilde{\boldsymbol{\eta}}^1_{u,h},\nabla\cdot\widetilde{\boldsymbol{e}}^1_{u,h}) \\
&= \Delta t(c(\widetilde{\boldsymbol{u}}^1_{ht},\widetilde{\boldsymbol{u}}^1_{ht},\widetilde{\boldsymbol{e}}^1_{u,h}) - c(\widetilde{\boldsymbol{w}}^1_t,\widetilde{\boldsymbol{w}}^1_t,\widetilde{\boldsymbol{e}}^1_{u,h})) \\
&\quad + \Delta t(s_h(\widetilde{\boldsymbol{u}}^1_{ht},\widetilde{\boldsymbol{u}}^1_{ht},\widetilde{\boldsymbol{u}}^1_{ht},\widetilde{\boldsymbol{e}}^1_{u,h}) + s_h(\widetilde{\boldsymbol{u}}^1_{ht},\widetilde{\boldsymbol{e}}^1_{u,h},\widetilde{\boldsymbol{u}}^1_{ht},\widetilde{\boldsymbol{e}}^1_{u,h})) \\
&\quad - (\widetilde{\boldsymbol{\eta}}^1_{u,h} - \boldsymbol{\eta}^0_{u,h},\widetilde{\boldsymbol{e}}^1_{u,h}).
\end{aligned}
\tag{2.3.13}
$$

In the last step we used the special choice of the interpolant.

Due to $\widetilde{\boldsymbol{\eta}}^1_u + \widetilde{\boldsymbol{\xi}}^1_{u,h} = \boldsymbol{u}(t_1) - \widetilde{\boldsymbol{u}}^1_{ht}$ we calculate for the convective term using skew-symmetry and

$$
\begin{aligned}
&c(\boldsymbol{u}(t_1),\boldsymbol{u}(t_1),\widetilde{\boldsymbol{e}}^1_{u,h}) - c(\widetilde{\boldsymbol{u}}^1_{ht},\widetilde{\boldsymbol{u}}^1_{ht},\widetilde{\boldsymbol{e}}^1_{u,h}) \\
&= c(\widetilde{\boldsymbol{\xi}}^1_{u,h},\boldsymbol{u}(t_1),\widetilde{\boldsymbol{e}}^1_{u,h}) + c(\widetilde{\boldsymbol{u}}^1_{ht},\widetilde{\boldsymbol{\xi}}^1_{u,h},\widetilde{\boldsymbol{e}}^1_{u,h}) \\
&\quad + c(\widetilde{\boldsymbol{\eta}}^1_u,\boldsymbol{u}(t_1),\widetilde{\boldsymbol{e}}^1_{u,h}) + c(\widetilde{\boldsymbol{u}}^1_{ht},\widetilde{\boldsymbol{\eta}}^1_u,\widetilde{\boldsymbol{e}}^1_{u,h}) \\
&= c(\widetilde{\boldsymbol{\xi}}^1_{u,h},\boldsymbol{u}(t_1),\widetilde{\boldsymbol{e}}^1_{u,h}) + c(\boldsymbol{u}(t_1) - \widetilde{\boldsymbol{\eta}}^1_u - \widetilde{\boldsymbol{\xi}}^1_{u,h},\widetilde{\boldsymbol{\eta}}^1_{u,h},\widetilde{\boldsymbol{e}}^1_{u,h}) \\
&\quad + c(\widetilde{\boldsymbol{\eta}}^1_u,\boldsymbol{u}(t_1),\widetilde{\boldsymbol{e}}^1_{u,h}) + c(\boldsymbol{u}(t_1) - \widetilde{\boldsymbol{\eta}}^1_u - \widetilde{\boldsymbol{\xi}}^1_{u,h},\widetilde{\boldsymbol{\eta}}^1_u,\widetilde{\boldsymbol{e}}^1_{u,h})
\end{aligned}
$$

$$
\begin{aligned}
&= c(\widetilde{\boldsymbol{\eta}}_{u,h}^1 + \widetilde{\boldsymbol{\eta}}_u^1, \boldsymbol{u}(t_1), \widetilde{\boldsymbol{e}}_{u,h}^1) + c(\boldsymbol{u}(t_1), \widetilde{\boldsymbol{\eta}}_{u,h}^1 + \widetilde{\boldsymbol{\eta}}_u^1, \widetilde{\boldsymbol{e}}_{u,h}^1) + c(\widetilde{\boldsymbol{e}}_{u,h}^1, \boldsymbol{u}(t_1), \widetilde{\boldsymbol{e}}_{u,h}^1) \\
&\quad - c(\widetilde{\boldsymbol{\eta}}_u^1 + \widetilde{\boldsymbol{\eta}}_{u,h}^1, \widetilde{\boldsymbol{\eta}}_u^1 + \widetilde{\boldsymbol{\eta}}_{u,h}^1, \widetilde{\boldsymbol{e}}_{u,h}^1) + c(\widetilde{\boldsymbol{e}}_{u,h}^1, \widetilde{\boldsymbol{\eta}}_{u,h}^1 + \widetilde{\boldsymbol{\eta}}_u^1, \widetilde{\boldsymbol{e}}_{u,h}^1).
\end{aligned}
$$

These terms can be estimated as

$$
\begin{aligned}
c(\widetilde{\boldsymbol{\eta}}_{u,h}^1 + \widetilde{\boldsymbol{\eta}}_u^1, \boldsymbol{u}(t_1), \widetilde{\boldsymbol{e}}_{u,h}^1) &\le C\|\widetilde{\boldsymbol{\eta}}_{u,h}^1 + \widetilde{\boldsymbol{\eta}}_u^1\|_0 \|\boldsymbol{u}(t_1)\|_2 \|\widetilde{\boldsymbol{e}}_{u,h}^1\|_1 \\
&\le \frac{C}{\nu}\|\widetilde{\boldsymbol{\eta}}_{u,h}^1 + \widetilde{\boldsymbol{\eta}}_u^1\|_0^2 \|\boldsymbol{u}(t_1)\|_2^2 + \frac{\nu}{32}\|\widetilde{\boldsymbol{e}}_{u,h}^1\|_1^2 \\
c(\boldsymbol{u}(t_1), \widetilde{\boldsymbol{\eta}}_{u,h}^1 + \widetilde{\boldsymbol{\eta}}_u^1, \widetilde{\boldsymbol{e}}_{u,h}^1) &\le C\|\widetilde{\boldsymbol{\eta}}_{u,h}^1 + \widetilde{\boldsymbol{\eta}}_u^1\|_0 \|\boldsymbol{u}(t_1)\|_2 \|\widetilde{\boldsymbol{e}}_{u,h}^1\|_1 \\
&\le \frac{C}{\nu}\|\widetilde{\boldsymbol{\eta}}_{u,h}^1 + \widetilde{\boldsymbol{\eta}}_u^1\|_0^2 \|\boldsymbol{u}(t_1)\|_2^2 + \frac{\nu}{32}\|\widetilde{\boldsymbol{e}}_{u,h}^1\|_1^2 \\
c(\widetilde{\boldsymbol{e}}_{u,h}^1, \boldsymbol{u}(t_1), \widetilde{\boldsymbol{e}}_{u,h}^1) &\le C\|\widetilde{\boldsymbol{e}}_{u,h}^1\|_0 \|\boldsymbol{u}(t_1)\|_2 \|\widetilde{\boldsymbol{e}}_{u,h}^1\|_1 \\
&\le \frac{C}{\nu}\|\widetilde{\boldsymbol{e}}_{u,h}^1\|_0^2 \|\boldsymbol{u}(t_1)\|_2^2 + \frac{\nu}{16}\|\widetilde{\boldsymbol{e}}_{u,h}^1\|_1^2 \\
c(\widetilde{\boldsymbol{\eta}}_u^1 + \widetilde{\boldsymbol{\eta}}_{u,h}^1, \widetilde{\boldsymbol{\eta}}_u^1 + \widetilde{\boldsymbol{\eta}}_{u,h}^1, \widetilde{\boldsymbol{e}}_{u,h}^1) &\le C\|\widetilde{\boldsymbol{\eta}}_u^1 + \widetilde{\boldsymbol{\eta}}_{u,h}^1\|_1^2 \|\widetilde{\boldsymbol{e}}_{u,h}^1\|_1 \\
&\le \frac{C}{\nu}\|\widetilde{\boldsymbol{\eta}}_u^1 + \widetilde{\boldsymbol{\eta}}_{u,h}^1\|_1^4 + \frac{\nu}{16}\|\widetilde{\boldsymbol{e}}_{u,h}^1\|_1^2 \\
c(\widetilde{\boldsymbol{e}}_{u,h}^1, \widetilde{\boldsymbol{\eta}}_{u,h}^1 + \widetilde{\boldsymbol{\eta}}_u^1, \widetilde{\boldsymbol{e}}_{u,h}^1) &\le C\|\widetilde{\boldsymbol{e}}_{u,h}^1\|_0^{1/2} \|\widetilde{\boldsymbol{\eta}}_{u,h}^1 + \widetilde{\boldsymbol{\eta}}_u^1\|_1 \|\widetilde{\boldsymbol{e}}_{u,h}^1\|_1^{3/2} \\
&\le \frac{C}{\nu^3}\|\widetilde{\boldsymbol{e}}_{u,h}^1\|_0^2 \|\widetilde{\boldsymbol{\eta}}_{u,h}^1 + \widetilde{\boldsymbol{\eta}}_u^1\|_1^4 + \frac{\nu}{16}\|\widetilde{\boldsymbol{e}}_{u,h}^1\|_1^2.
\end{aligned}
$$

For the last estimate we used the inequality

$$
\begin{aligned}
a^{1/2}b^{3/2} &= (\nu^{-3/2}a)^{1/2}(\sqrt{\nu}b)^{3/2} \le C(((\nu^{-3/2}a)^{1/2})^4 + ((\sqrt{\nu}b)^{3/2})^{4/3}) \\
&\le C(\nu^{-3}a^2 + \nu b^2).
\end{aligned}
$$

In combination, the error with respect to the convective terms is given by

$$
\begin{aligned}
&c(\boldsymbol{u}(t_1), \boldsymbol{u}(t_1), \widetilde{\boldsymbol{e}}_{u,h}^1) - c(\widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{e}}_{u,h}^1) \\
&\le C\|\widetilde{\boldsymbol{e}}_{u,h}^1\|_0^2 \left(\frac{\|\widetilde{\boldsymbol{\eta}}_u^1\|_1^4 + \|\widetilde{\boldsymbol{\eta}}_{u,h}^1\|_1^4}{\nu^3} + \frac{\|\boldsymbol{u}(t_1)\|_2^2}{\nu} \right) + \frac{\nu}{4}\|\widetilde{\boldsymbol{e}}_{u,h}^1\|_1^2 \\
&\quad + \frac{C}{\nu}(\|\widetilde{\boldsymbol{\eta}}_u^1\|_1^4 + \|\widetilde{\boldsymbol{\eta}}_{u,h}^1\|_1^4 + \|\widetilde{\boldsymbol{\eta}}_{u,h}^1\|_0^2 \|\boldsymbol{u}(t_1)\|_2^2).
\end{aligned}
$$

For the nonlinear stabilization we again use $\widetilde{\boldsymbol{w}}_{ht}^1 = \boldsymbol{u}(t_1) - (\widetilde{\boldsymbol{\eta}}_u^1 + \widetilde{\boldsymbol{\eta}}_{u,h}^1)$ and obtain

$$
\begin{aligned}
&s_h(\widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{w}}_{ht}^1, \widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{e}}_{u,h}^1) = s_h(\widetilde{\boldsymbol{u}}_{ht}^1, \boldsymbol{u}(t_1), \widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{e}}_{u,h}^1) - s_h(\widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{\eta}}_u^1 + \widetilde{\boldsymbol{\eta}}_{u,h}^1, \widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{e}}_{u,h}^1) \\
&\le C \sum_{M \in \mathcal{M}_h} \tau_M^1 |\widetilde{\boldsymbol{u}}_M^1|^2 \|\kappa_M(\nabla \boldsymbol{u}(t_1))\|_{0,M}^2 + C \sum_{M \in \mathcal{M}_h} \tau_M^1 |\widetilde{\boldsymbol{u}}_M^1|^2 \|\nabla(\widetilde{\boldsymbol{\eta}}_u^1 + \widetilde{\boldsymbol{\eta}}_{u,h}^1)\|_{0,M}^2 \\
&\quad + \frac{1}{4} \sum_{M \in \mathcal{M}_h} \tau_M^1 \|\kappa_M((\widetilde{\boldsymbol{u}}_M^1 \cdot \nabla)\widetilde{\boldsymbol{e}}_{u,h}^1)\|_{0,M}^2
\end{aligned}
$$

$$\leq C \max_{M\in\mathcal{M}_h}\{\tau_M^1|\widetilde{\boldsymbol{u}}_M^1|^2\}\left(\|\nabla(\widetilde{\boldsymbol{\eta}}_u^1+\widetilde{\boldsymbol{\eta}}_{u,h}^1)\|_0^2+\sum_{M\in\mathcal{M}_h}\|\kappa_M(\nabla\boldsymbol{u}(t_1))\|_{0,M}^2\right)$$
$$+\frac{1}{4}\sum_{M\in\mathcal{M}_h}\tau_M^1\|\kappa_M((\widetilde{\boldsymbol{u}}_M^1\cdot\nabla)\widetilde{\boldsymbol{e}}_{u,h}^1)\|_{0,M}^2.$$

We summarize the estimates in

$$\begin{aligned}
&C\left(1-\frac{\Delta t}{\nu}\|\boldsymbol{u}(t_1)\|_2^2-\Delta t\frac{\|\widetilde{\boldsymbol{\eta}}_u^1\|_1^4+\|\widetilde{\boldsymbol{\eta}}_{u,h}^1\|_1^4}{\nu^3}\right)\|\widetilde{\boldsymbol{e}}_{u,h}^1\|_0^2+\nu\Delta t\|\nabla\widetilde{\boldsymbol{e}}_{u,h}^1\|_0^2\\
&\quad+\gamma\Delta t\|\nabla\cdot\widetilde{\boldsymbol{e}}_{u,h}^1\|_0^2+2\Delta t\sum_{M\in\mathcal{M}_h}\tau_M^1\|\kappa_M((\widetilde{\boldsymbol{u}}_{ht}^1\cdot\nabla)\widetilde{\boldsymbol{e}}_{u,h}^1)\|_{0,M}^2\\
&\leq C\Delta t\max_{M\in\mathcal{M}_h}\{\tau_M^1|\widetilde{\boldsymbol{u}}_M^1|^2\}\left(\|\widetilde{\boldsymbol{\eta}}_u^1\|_1^2+\|\nabla\widetilde{\boldsymbol{\eta}}_{u,h}^1\|_0^2+\sum_{M\in\mathcal{M}_h}\|\kappa_M(\nabla\boldsymbol{u}(t_1))\|_{0,M}^2\right)\\
&\quad+C\|\widetilde{\boldsymbol{\eta}}_{u,h}^1-\boldsymbol{\eta}_{u,h}^0\|_0^2+C\frac{\Delta t}{\nu}(\|\widetilde{\boldsymbol{\eta}}_u^1\|_1^4+\|\widetilde{\boldsymbol{\eta}}_{u,h}^1\|_1^4+\|\widetilde{\boldsymbol{\eta}}_{u,h}^1\|_0^2\|\boldsymbol{u}(t_1)\|_2^2)
\end{aligned}\tag{2.3.14}$$

due to

$$(\widetilde{\boldsymbol{\eta}}_{u,h}^1-\boldsymbol{\eta}_{u,h}^0,\widetilde{\boldsymbol{e}}_{u,h}^1)\leq\frac{1}{4}\|\widetilde{\boldsymbol{e}}_{u,h}^1\|_0^2+C\|\widetilde{\boldsymbol{\eta}}_{u,h}^1-\boldsymbol{\eta}_{u,h}^0\|_0^2.$$

Finally, we obtain:

$$\begin{aligned}
&C\,(1-K)\,\|\widetilde{\boldsymbol{e}}_{u,h}^1\|_0^2+\nu\Delta t\|\nabla\widetilde{\boldsymbol{e}}_{u,h}^1\|_0^2\\
&\quad+\gamma\Delta t\|\nabla\cdot\widetilde{\boldsymbol{e}}_{u,h}^1\|_0^2+2\Delta t\sum_{M\in\mathcal{M}_h}\tau_M^1\|\kappa_M((\widetilde{\boldsymbol{u}}_{ht}^1\cdot\nabla)\widetilde{\boldsymbol{e}}_{u,h}^1)\|_{0,M}^2\\
&\lesssim\left((\nu+\gamma)h^{2k_u+2}+\gamma^{-1}h^{2k_p+4}+\Delta t^2h^2\left(\frac{\nu+\gamma}{\nu}+\frac{1}{\nu\gamma}\right)\right)\left(1+\frac{\Delta t}{\nu}\right)\\
&\quad+\Delta t\frac{(\nu+\gamma)^2h^{4k_u}+\gamma^{-2}h^{4k_p+4}+\Delta t^4(\nu+\gamma+\gamma^{-1})^2}{\nu^3}\\
&\quad+\Delta t\max_{M\in\mathcal{M}_h}\{\tau_M^1|\widetilde{\boldsymbol{u}}_M^1|^2\}\left(\frac{\nu+\gamma}{\nu}h^{2k_u}+\frac{h^{2k_p+2}}{\nu\gamma}+\Delta t^2\frac{\nu+\gamma+\gamma^{-1}}{\nu}+h^{2s}\right).
\end{aligned}$$

For an estimate on the gradient of the pressure we test the projection error equation with $\nabla(\xi_{p,h}^1-\xi_{p,h}^0)$ to obtain

$$\begin{aligned}
&\frac{2\Delta t}{3}\|\nabla(\xi_{p,h}^1-\xi_{p,h}^0)\|_0\\
&=\frac{2\Delta t}{3}\frac{\|\nabla(\xi_{p,h}^1-\xi_{p,h}^0)\|_0^2}{\|\nabla(\xi_{p,h}^1-\xi_{p,h}^0)\|_0}=\frac{(\boldsymbol{\xi}_{u,h}-\widetilde{\boldsymbol{\xi}}_{u,h},\nabla(\xi_{p,h}^1-\xi_{p,h}^0))}{\|\nabla(\xi_{p,h}^1-\xi_{p,h}^0)\|_0}\\
&\leq\|\boldsymbol{\xi}_{u,h}-\widetilde{\boldsymbol{\xi}}_{u,h}\|_0\leq\|\boldsymbol{e}_{u,h}-\widetilde{\boldsymbol{e}}_{u,h}\|_0+\|\boldsymbol{\eta}_{u,h}-\widetilde{\boldsymbol{\eta}}_{u,h}\|_0\\
\Rightarrow&(\Delta t)^2\|\nabla(\xi_{p,h}^1-\xi_{p,h}^0)\|_0^2\leq\|\widetilde{\boldsymbol{e}}_{u,h}\|_0^2+\|\widetilde{\boldsymbol{\eta}}_{u,h}\|_0^2.
\end{aligned}\tag{2.3.15}$$

□

2.3.4 Discretization Error after Initialization

Now, we are in position to state the error bounds also for $n \geq 2$.

Lemma 2.3.2. *For all $1 \leq m \leq N$ the discretization error due to spatial discretization can be bounded as*

$$\begin{aligned}
&\|\widetilde{\boldsymbol{e}}_{u,h}^m\|_0^2 + \|2\boldsymbol{e}_{u,h}^m - \boldsymbol{e}_{u,h}^{m-1}\|_0^2 + \frac{4}{3}(\Delta t)^2\|\nabla\xi_{p,h}^m\|_0^2 \\
&\quad + \sum_{n=2}^{m}\Big(\Delta t\nu\|\nabla\widetilde{\boldsymbol{e}}_{u,h}^n\|_0^2 + \Delta t\gamma\|\nabla\cdot\widetilde{\boldsymbol{e}}_{u,h}^n\|_0^2 + \|\delta_{tt}\boldsymbol{e}_{u,h}^n\|_0^2 \\
&\qquad\qquad + 2\Delta t\sum_{M\in\mathcal{M}_h}\tau_M^n\|\kappa_M((\widetilde{\boldsymbol{u}}_M^n\cdot\nabla)\widetilde{\boldsymbol{e}}_{u,h}^n)\|_{0,M}^2\Big) \\
&\leq C_{G,h}\left((\nu+\gamma)h^{2k_u+2} + \frac{h^{2k_p+4}}{\gamma} + \Delta t^2 h^2\left(\frac{\nu+\gamma}{\nu} + \frac{1}{\nu\gamma}\right)\right)\left(\frac{1}{\Delta t} + \frac{1}{\nu}\right) \\
&\quad + \frac{C_{G,h}}{\nu^3}\left((\nu+\gamma)^2 h^{4k_u} + \gamma^{-2}h^{4k_p+4} + \Delta t^4(\nu+\gamma+\gamma^{-1})^2\right) \qquad (2.3.16) \\
&\quad + C_{G,h}\max_{M\in\mathcal{M}_h}\{\tau_M^1|\widetilde{\boldsymbol{u}}_M^1|^2\}\left(\frac{\nu+\gamma}{\nu}h^{2k_u} + \frac{h^{2k_p+2}}{\nu\gamma} + \Delta t^2\frac{\nu+\gamma+\gamma^{-1}}{\nu} + h^{2s}\right) \\
&\quad + \frac{C_{G,h}}{\nu\gamma}\left((\nu+\gamma)h^{2k_u} + \gamma^{-1}h^{2k_p+2} + \Delta t^2(\nu+\gamma+\gamma^{-1})\right).
\end{aligned}$$

The Gronwall term $C_{G,h}$ behaves like

$$C_{G,h} \sim \exp\left(\frac{T}{1-K}\right) \qquad (2.3.17)$$

where

$$\begin{aligned}
K &:= \frac{1}{4} + C\frac{\Delta t}{\nu} + C\Delta t\frac{\|\widetilde{\boldsymbol{\eta}}_u^n\|_1^4 + \|\widetilde{\boldsymbol{\eta}}_{u,h}^n\|_1^4}{\nu^3} \qquad (2.3.18) \\
&\leq \frac{1}{4} + C\left(\frac{\Delta t}{\nu} + \Delta t\frac{(\nu+\gamma)^2h^{4k_u} + \gamma^{-2}h^{4k_p+4} + (\Delta t)^4(\nu+\gamma+\gamma^{-1})^2}{\nu^5}\right)
\end{aligned}$$

and $K < 1$ is required.

Proof. Subtracting the convection-diffusion and projection equations for $\widetilde{\boldsymbol{w}}_t^n$ and $\widetilde{\boldsymbol{u}}_{ht}^n$ from each other gives for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $\phi_h \in Q_h$

$$\begin{aligned}
&\left(\frac{3\widetilde{\boldsymbol{e}}_{u,h}^n - 4\boldsymbol{e}_{u,h}^{n-1} + \boldsymbol{e}_{u,h}^{n-2}}{2\Delta t}, \boldsymbol{v}_h\right) + \nu(\nabla\widetilde{\boldsymbol{e}}_{u,h}^n, \nabla\boldsymbol{v}_h) + \gamma(\nabla\cdot\widetilde{\boldsymbol{e}}_{u,h}^n, \nabla\cdot\boldsymbol{v}_h) \\
&\quad + (\nabla\xi_{p,h}^{n-1}, \boldsymbol{v}_h) + c(\widetilde{\boldsymbol{w}}_t^n, \widetilde{\boldsymbol{w}}_t^n, \boldsymbol{v}_h) - c(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) \ - s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) \\
&= -\left(\frac{3\widetilde{\boldsymbol{\eta}}_{u,h}^n - 4\boldsymbol{\eta}_{u,h}^{n-1} + \boldsymbol{\eta}_{u,h}^{n-2}}{2\Delta t}, \boldsymbol{v}_h\right) - \nu(\nabla\widetilde{\boldsymbol{\eta}}_{u,h}^n, \nabla\boldsymbol{v}_h) - \gamma(\nabla\cdot\widetilde{\boldsymbol{\eta}}_{u,h}^n, \nabla\cdot\boldsymbol{v}_h)
\end{aligned}$$

(2.3.19)

and

$$\left(\frac{3\boldsymbol{\xi}^n_{u,h} - 3\widetilde{\boldsymbol{\xi}}^n_{u,h}}{2\Delta t} + \nabla(\xi^n_{p,h} - \xi^{n-1}_{p,h}), \nabla \phi_h \right) = 0. \tag{2.3.20}$$

We first test (2.3.19) with $4\Delta t \boldsymbol{e}^n_{u,h}$ to get

$$\begin{aligned}
&2\left(3\widetilde{\boldsymbol{e}}^n_{u,h} - 4\boldsymbol{e}^{n-1}_{u,h} + \boldsymbol{e}^{n-2}_{u,h}, \widetilde{\boldsymbol{e}}^n_{u,h}\right) + 4\Delta t \nu(\nabla \widetilde{\boldsymbol{e}}^n_{u,h}, \nabla \widetilde{\boldsymbol{e}}^n_{u,h}) \\
&\quad + 4\Delta t s_h(\widetilde{\boldsymbol{u}}^n_{ht}, \widetilde{\boldsymbol{e}}^n_{u,h}, \widetilde{\boldsymbol{u}}^n_{ht}, \widetilde{\boldsymbol{e}}^n_{u,h}) \\
&= -4\Delta t \left(c(\widetilde{\boldsymbol{w}}^n_t, \widetilde{\boldsymbol{w}}^n_t, \widetilde{\boldsymbol{e}}^n_{u,h}) - c(\widetilde{\boldsymbol{u}}^n_{ht}, \widetilde{\boldsymbol{u}}^n_{ht}, \widetilde{\boldsymbol{e}}^n_{u,h})\right) \\
&\quad + 4\Delta t \left(s_h(\widetilde{\boldsymbol{u}}^n_{ht}, \widetilde{\boldsymbol{u}}^n_{ht}, \widetilde{\boldsymbol{u}}^n_{ht}, \widetilde{\boldsymbol{e}}^n_{u,h}) + s_h(\widetilde{\boldsymbol{u}}^n_{ht}, \widetilde{\boldsymbol{e}}^n_{u,h}, \widetilde{\boldsymbol{u}}^n_{ht}, \widetilde{\boldsymbol{e}}^n_{u,h})\right) \\
&\quad - \left(\frac{3\widetilde{\boldsymbol{\eta}}^n_{u,h} - 4\boldsymbol{\eta}^{n-1}_{u,h} + \boldsymbol{\eta}^{n-2}_{u,h}}{2\Delta t}, \widetilde{\boldsymbol{e}}_h\right) - \nu(\nabla \widetilde{\boldsymbol{\eta}}^n_{u,h}, \nabla \widetilde{\boldsymbol{e}}_h) \\
&\quad - \gamma(\nabla \cdot \widetilde{\boldsymbol{\eta}}^n_{u,h}, \nabla \cdot \widetilde{\boldsymbol{e}}_h) - 4\Delta t(\nabla \xi^{n-1}_{p,h}, \widetilde{\boldsymbol{e}}^n_{u,h}) \\
&= -4\Delta t \left(c(\widetilde{\boldsymbol{w}}^n_t, \widetilde{\boldsymbol{w}}^n_t, \widetilde{\boldsymbol{e}}^n_{u,h}) - c(\widetilde{\boldsymbol{u}}^n_{ht}, \widetilde{\boldsymbol{u}}^n_{ht}, \widetilde{\boldsymbol{e}}^n_{u,h})\right) \\
&\quad + 4\Delta t \left(s_h(\widetilde{\boldsymbol{u}}^n_{ht}, \widetilde{\boldsymbol{u}}^n_{ht}, \widetilde{\boldsymbol{u}}^n_{ht}, \widetilde{\boldsymbol{e}}^n_{u,h}) + s_h(\widetilde{\boldsymbol{u}}^n_{ht}, \widetilde{\boldsymbol{e}}^n_{u,h}, \widetilde{\boldsymbol{u}}^n_{ht}, \widetilde{\boldsymbol{e}}^n_{u,h})\right) \\
&\quad - \left(\frac{3\widetilde{\boldsymbol{\eta}}^n_{u,h} - 4\boldsymbol{\eta}^{n-1}_{u,h} + \boldsymbol{\eta}^{n-2}_{u,h}}{2\Delta t}, \widetilde{\boldsymbol{e}}_h\right) - 4\Delta t(\nabla e^{n-1}_{p,h}, \widetilde{\boldsymbol{e}}^n_{u,h})
\end{aligned} \tag{2.3.21}$$

where the last step follows from the choice for the interpolation operator. The last pressure term in the above equation can be bounded by

$$\begin{aligned}
(\nabla e^{n-1}_{p,h}, \widetilde{\boldsymbol{e}}^n_{u,h}) &= (\nabla e^{n-1}_{p,h}, \widetilde{\boldsymbol{\xi}}^n_{u,h}) = \frac{2\Delta t}{3}(\nabla(\xi^n_{p,h} - \xi^{n-1}_{p,h}), \nabla e^{n-1}_{p,h}) \\
&= \frac{2\Delta t}{3}(\nabla(\xi^n_{p,h} - \xi^{n-1}_{p,h}), \nabla(\xi^{n-1}_{p,h} - \eta^{n-1}_{p,h})) \\
&= \frac{\Delta t}{3}(\|\nabla \xi^n_{p,h}\|_0^2 - \|\nabla(\xi^n_{p,h} - \xi^{n-1}_{p,h})\|_0^2 - \|\nabla \xi^{n-1}_{p,h}\|_0^2) \\
&\quad - \frac{2\Delta t}{3}(\nabla(\xi^n_{p,h} - \xi^{n-1}_{p,h}), \nabla \eta^{n-1}_{p,h}) \\
&= \frac{\Delta t}{3}(\|\nabla \xi^n_{p,h}\|_0^2 - \|\nabla \xi^{n-1}_{p,h}\|_0^2) - \frac{3}{4\Delta t}\|\widetilde{\boldsymbol{e}}^n_{u,h} - \boldsymbol{e}^n_{u,h}\|_0^2 \\
&\quad + (\nabla \cdot \widetilde{\boldsymbol{e}}^n_{u,h}, \eta^{n-1}_{p,h}) \\
&\geq \frac{\Delta t}{3}(\|\nabla \xi^n_{p,h}\|_0^2 - \|\nabla \xi^{n-1}_{p,h}\|_0^2) - \frac{3}{4\Delta t}\|\widetilde{\boldsymbol{e}}^n_{u,h} - \boldsymbol{e}^n_{u,h}\|_0^2 \\
&\quad - \frac{\gamma}{4}\|\nabla \cdot \widetilde{\boldsymbol{e}}^n_{u,h}\|_0^2 - \frac{C}{\gamma}\|\eta^{n-1}_{p,h}\|_0^2.
\end{aligned}$$

A splitting (cf. (A.1.1)) of the time derivative term gives

$$\begin{aligned}
&(2(3\widetilde{\boldsymbol{e}}^n_{u,h} - 4\boldsymbol{e}^{n-1}_{u,h} + \boldsymbol{e}^{n-2}_{u,h}), \widetilde{\boldsymbol{e}}^n_{u,h}) = I_1 + I_2 + I_3 \\
&= 3\|\widetilde{\boldsymbol{e}}^n_{u,h}\|_0^2 + 3\|\boldsymbol{e}^n_{u,h} - \widetilde{\boldsymbol{e}}^n\|_0^2 - 2\|\boldsymbol{e}^n_{u,h}\|_0^2 + \|2\boldsymbol{e}^n_{u,h} - \boldsymbol{e}^{n-1}_{u,h}\|_0^2
\end{aligned}$$

$$+ \|\delta_{tt}\boldsymbol{e}_{u,h}^n\|_0^2 - \|\boldsymbol{e}^{n-1}\|_0^2 - \|2\boldsymbol{e}_{u,h}^{n-1} - \boldsymbol{e}_{u,h}^{n-2}\|_0^2.$$

The second term I_2 vanishes due to the fact that $\boldsymbol{e}_{u,h}^n$ is weakly divergence-free.

With the identity $(a-b,b) = \frac{1}{2}(\|a\|_0^2 - \|a-b\|_0^2 - \|b\|_0^2)$ and $\|\boldsymbol{e}_{u,h}^n\| \overset{(1.2.1)}{\leq} \|\widetilde{\boldsymbol{e}}_{u,h}^n\|$ we have so far

$$\begin{aligned}
&\|\widetilde{\boldsymbol{e}}_{u,h}^n\|_0^2 + \|2\boldsymbol{e}_{u,h}^n - \boldsymbol{e}_{u,h}^{n-1}\|_0^2 + \frac{4}{3}(\Delta t)^2\|\nabla\xi_{p,h}^n\|_0^2 \\
&\quad + 4\Delta t\nu\|\nabla\widetilde{\boldsymbol{e}}_{u,h}^n\|_0^2 + 3\Delta t\gamma\|\nabla\cdot\widetilde{\boldsymbol{e}}_{u,h}^n\|_0^2 + \|\delta_{tt}\boldsymbol{e}_{u,h}^n\|_0^2 \\
&\quad + 4\Delta t\|\kappa_M((\widetilde{\boldsymbol{u}}_{ht}^n\cdot\nabla)\widetilde{\boldsymbol{e}}_{u,h}^n)\|_0^2 \\
&\leq \|\widetilde{\boldsymbol{e}}_{u,h}^{n-1}\|_0^2 + \|2\boldsymbol{e}_{u,h}^{n-1} - \boldsymbol{e}_{u,h}^{n-2}\|_0^2 + \frac{4}{3}(\Delta t)^2\|\nabla\xi_{p,h}^{n-1}\|_0^2 \\
&\quad - 4\Delta t\left(c(\widetilde{\boldsymbol{w}}_t^n,\widetilde{\boldsymbol{w}}_t^n,\widetilde{\boldsymbol{e}}_{u,h}^n) - c(\widetilde{\boldsymbol{u}}_{ht}^n,\widetilde{\boldsymbol{u}}_{ht}^n,\widetilde{\boldsymbol{e}}_{u,h}^n)\right) \\
&\quad + 4\Delta t\left(s_h(\widetilde{\boldsymbol{u}}_{ht}^n,\widetilde{\boldsymbol{w}}_t^n,\widetilde{\boldsymbol{u}}_{ht}^n,\widetilde{\boldsymbol{e}}_{u,h}^n) - s_h(\widetilde{\boldsymbol{u}}_{ht}^n,\widetilde{\boldsymbol{\eta}}_{u,h}^n,\widetilde{\boldsymbol{u}}_{ht}^n,\widetilde{\boldsymbol{e}}_{u,h}^n)\right) \\
&\quad - \left(\frac{3\widetilde{\boldsymbol{\eta}}_{u,h}^n - 4\boldsymbol{\eta}_{u,h}^{n-1} + \boldsymbol{\eta}_{u,h}^{n-2}}{2\Delta t}, \widetilde{\boldsymbol{e}}_{u,h}\right) + \frac{C}{\gamma}\Delta t\|\eta_{p,h}^{n-1}\|_0^2.
\end{aligned} \tag{2.3.22}$$

For the terms on the right-hand side containing approximations errors we use the estimate

$$(3\widetilde{\boldsymbol{\eta}}_{u,h}^n - 4\boldsymbol{\eta}_{u,h}^{n-1} + \boldsymbol{\eta}_{u,h}^{n-2}, \widetilde{\boldsymbol{e}}_{u,h}^n) \leq C(\|\widetilde{\boldsymbol{\eta}}_{u,h}^n\|_0^2 + \|\boldsymbol{\eta}_{u,h}^{n-1}\|_0^2 + \|\boldsymbol{\eta}_{u,h}^{n-2}\|_0^2) + \frac{1}{4}\|\widetilde{\boldsymbol{e}}_{u,h}^n\|_0^2$$

Due to $\widetilde{\boldsymbol{\eta}}_u^n + \widetilde{\boldsymbol{\xi}}_{u,h}^n = \boldsymbol{u}(t_n) - \widetilde{\boldsymbol{u}}_{ht}^n$ we calculate for the convective term using skew-symmetry and

$$\begin{aligned}
&c(\boldsymbol{u}(t_n),\boldsymbol{u}(t_n),\widetilde{\boldsymbol{e}}_{u,h}^n) - c(\widetilde{\boldsymbol{u}}_{ht}^n,\widetilde{\boldsymbol{u}}_{ht}^n,\widetilde{\boldsymbol{e}}_{u,h}^n) \\
&= c(\widetilde{\boldsymbol{\xi}}_{u,h}^n,\boldsymbol{u}(t_n),\widetilde{\boldsymbol{e}}_{u,h}^n) + c(\widetilde{\boldsymbol{u}}_{ht}^n,\widetilde{\boldsymbol{\xi}}_{u,h}^n,\widetilde{\boldsymbol{e}}_{u,h}^n) + \\
&\quad c(\widetilde{\boldsymbol{\eta}}_u^n,\boldsymbol{u}(t_n),\widetilde{\boldsymbol{e}}_{u,h}^n) + c(\widetilde{\boldsymbol{u}}_{ht}^n,\widetilde{\boldsymbol{\eta}}_u^n,\widetilde{\boldsymbol{e}}_{u,h}^n) \\
&= c(\widetilde{\boldsymbol{\xi}}_{u,h}^n,\boldsymbol{u}(t_n),\widetilde{\boldsymbol{e}}_{u,h}^n) + c(\boldsymbol{u}(t_n) - \widetilde{\boldsymbol{\eta}}_u^n - \widetilde{\boldsymbol{\xi}}_{u,h}^n,\widetilde{\boldsymbol{\eta}}_{u,h}^n,\widetilde{\boldsymbol{e}}_{u,h}^n) \\
&\quad + c(\widetilde{\boldsymbol{\eta}}_u^n,\boldsymbol{u}(t_n),\widetilde{\boldsymbol{e}}_{u,h}^n) + c(\boldsymbol{u}(t_n) - \widetilde{\boldsymbol{\eta}}_u^n - \widetilde{\boldsymbol{\xi}}_{u,h}^n,\widetilde{\boldsymbol{\eta}}_u^n,\widetilde{\boldsymbol{e}}_{u,h}^n) \\
&= c(\widetilde{\boldsymbol{\eta}}_{u,h}^n + \widetilde{\boldsymbol{\eta}}_u^n,\boldsymbol{u}(t_n),\widetilde{\boldsymbol{e}}_{u,h}^n) + c(\boldsymbol{u}(t_n),\widetilde{\boldsymbol{\eta}}_{u,h}^n + \widetilde{\boldsymbol{\eta}}_u^n,\widetilde{\boldsymbol{e}}_{u,h}^n) \\
&\quad + c(\widetilde{\boldsymbol{e}}_{u,h}^n,\boldsymbol{u}(t_n),\widetilde{\boldsymbol{e}}_{u,h}^n) - c(\widetilde{\boldsymbol{\eta}}_u^n + \widetilde{\boldsymbol{\eta}}_{u,h}^n,\widetilde{\boldsymbol{\eta}}_u^n + \widetilde{\boldsymbol{\eta}}_{u,h}^n,\widetilde{\boldsymbol{e}}_{u,h}^n) \\
&\quad + c(\widetilde{\boldsymbol{e}}_{u,h}^n,\widetilde{\boldsymbol{\eta}}_{u,h}^n + \widetilde{\boldsymbol{\eta}}_u^n,\widetilde{\boldsymbol{e}}_{u,h}^n).
\end{aligned} \tag{2.3.23}$$

These terms can be estimated as

$$\begin{aligned}
c(\widetilde{\boldsymbol{\eta}}_{u,h}^n + \widetilde{\boldsymbol{\eta}}_u^n,\boldsymbol{u}(t_n),\widetilde{\boldsymbol{e}}_{u,h}^n) &\leq C\|\widetilde{\boldsymbol{\eta}}_{u,h}^n + \widetilde{\boldsymbol{\eta}}_u^n\|_0\|\boldsymbol{u}(t_n)\|_2\|\widetilde{\boldsymbol{e}}_{u,h}^n\|_1 \\
&\leq \frac{C}{\nu}\|\widetilde{\boldsymbol{\eta}}_{u,h}^n + \widetilde{\boldsymbol{\eta}}_u^n\|_0^2\|\boldsymbol{u}(t_n)\|_2^2 + \frac{\nu}{32}\|\widetilde{\boldsymbol{e}}_{u,h}^n\|_1^2 \\
c(\boldsymbol{u}(t_n),\widetilde{\boldsymbol{\eta}}_{u,h}^n + \widetilde{\boldsymbol{\eta}}_u^n,\widetilde{\boldsymbol{e}}_{u,h}^n) &\leq C\|\widetilde{\boldsymbol{\eta}}_{u,h}^n + \widetilde{\boldsymbol{\eta}}_u^n\|_0\|\boldsymbol{u}(t_n)\|_2\|\widetilde{\boldsymbol{e}}_{u,h}^n\|_1 \\
&\leq \frac{C}{\nu}\|\widetilde{\boldsymbol{\eta}}_{u,h}^n + \widetilde{\boldsymbol{\eta}}_u^n\|_0^2\|\boldsymbol{u}(t_n)\|_2^2 + \frac{\nu}{32}\|\widetilde{\boldsymbol{e}}_{u,h}^n\|_1^2
\end{aligned}$$

$$\begin{aligned}
c(\widetilde{\boldsymbol{e}}_{u,h}^n, \boldsymbol{u}(t_n), \widetilde{\boldsymbol{e}}_{u,h}^n) &\le C\|\widetilde{\boldsymbol{e}}_{u,h}^n\|_0 \|\boldsymbol{u}(t_n)\|_2 \|\widetilde{\boldsymbol{e}}_{u,h}^n\|_1 \\
&\le \frac{C}{\nu}\|\widetilde{\boldsymbol{e}}_{u,h}^n\|_0^2 \|\boldsymbol{u}(t_n)\|_2^2 + \frac{\nu}{16}\|\widetilde{\boldsymbol{e}}_{u,h}^n\|_1^2 \\
c(\widetilde{\boldsymbol{\eta}}_u^n + \widetilde{\boldsymbol{\eta}}_{u,h}^n, \widetilde{\boldsymbol{\eta}}_u^n + \widetilde{\boldsymbol{\eta}}_{u,h}^n, \widetilde{\boldsymbol{e}}_{u,h}^n) &\le C\|\widetilde{\boldsymbol{\eta}}_u^n + \widetilde{\boldsymbol{\eta}}_{u,h}^n\|_1^2 \|\widetilde{\boldsymbol{e}}_{u,h}^n\|_1 \\
&\le \frac{C}{\nu}\|\widetilde{\boldsymbol{\eta}}_u^n + \widetilde{\boldsymbol{\eta}}_{u,h}^n\|_1^4 + \frac{\nu}{16}\|\widetilde{\boldsymbol{e}}_{u,h}^n\|_1^2 \\
c(\widetilde{\boldsymbol{e}}_{u,h}^n, \widetilde{\boldsymbol{\eta}}_{u,h}^n + \widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{e}}_{u,h}^n) &\le C\|\widetilde{\boldsymbol{e}}_{u,h}^n\|_0^{1/2} \|\widetilde{\boldsymbol{\eta}}_{u,h}^n + \widetilde{\boldsymbol{\eta}}_u^n\|_1 \|\widetilde{\boldsymbol{e}}_{u,h}^n\|_1^{3/2} \\
&\le \frac{C}{\nu^3}\|\widetilde{\boldsymbol{e}}_{u,h}^n\|_0^2 \|\widetilde{\boldsymbol{\eta}}_{u,h}^n + \widetilde{\boldsymbol{\eta}}_u^n\|_1^4 + \frac{\nu}{16}\|\widetilde{\boldsymbol{e}}_{u,h}^n\|_1^2.
\end{aligned}$$

For the last estimate we used the inequality

$$\begin{aligned}
a^{1/2}b^{3/2} &= (\nu^{-3/2}a)^{1/2}(\sqrt{\nu}b)^{3/2} \le C(((\nu^{-3/2}a)^{1/2})^4 + ((\sqrt{\nu}b)^{3/2})^{4/3}) \\
&\le C(\nu^{-3}a^2 + \nu b^2).
\end{aligned}$$

In combination, the error with respect to the convective terms is given by

$$\begin{aligned}
&c(\boldsymbol{u}(t_n), \boldsymbol{u}(t_n), \widetilde{\boldsymbol{e}}_{u,h}^n) - c(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{e}}_{u,h}^n) \\
&\le C\|\widetilde{\boldsymbol{e}}_{u,h}^n\|_0^2 \left(\frac{\|\widetilde{\boldsymbol{\eta}}_u^n\|_1^4 + \|\widetilde{\boldsymbol{\eta}}_{u,h}^n\|_1^4}{\nu^3} + \frac{\|\boldsymbol{u}(t_n)\|_2^2}{\nu} \right) + \frac{\nu}{4}\|\widetilde{\boldsymbol{e}}_{u,h}^n\|_1^2 \\
&\quad + \frac{C}{\nu}(\|\widetilde{\boldsymbol{\eta}}_u^n\|_1^4 + \|\widetilde{\boldsymbol{\eta}}_{u,h}^n\|_1^4 + \|\widetilde{\boldsymbol{\eta}}_{u,h}^n\|_0^2 \|\boldsymbol{u}(t_n)\|_2^2).
\end{aligned}$$

For the nonlinear stabilization we again use $\widetilde{\boldsymbol{w}}_{ht}^n = \boldsymbol{u}(t_n) - (\widetilde{\boldsymbol{\eta}}_u^n + \widetilde{\boldsymbol{\eta}}_{u,h}^n)$ and obtain

$$\begin{aligned}
&s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{w}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{e}}_{u,h}^n) \\
&= s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{u}(t_n), \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{e}}_{u,h}^n) - s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{\eta}}_u^n + \widetilde{\boldsymbol{\eta}}_{u,h}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{e}}_{u,h}^n) \\
&\le C \sum_{M\in\mathcal{M}_h} \tau_M^n |\widetilde{\boldsymbol{u}}_M^n|^2 \left(\|\kappa_M(\nabla \boldsymbol{u}(t_n))\|_{0,M}^2 + \|\nabla(\widetilde{\boldsymbol{\eta}}_u^n + \widetilde{\boldsymbol{\eta}}_{u,h}^n)\|_{0,M}^2 \right) \\
&\quad + \frac{1}{4} \sum_{M\in\mathcal{M}_h} \tau_M^n \|\kappa_M((\widetilde{\boldsymbol{u}}_M^n \cdot \nabla)\widetilde{\boldsymbol{e}}_{u,h}^n)\|_{0,M}^2 \\
&\le C \max_{M\in\mathcal{M}_h} \{\tau_M^n |\widetilde{\boldsymbol{u}}_M^n|^2\} \left(\|\nabla(\widetilde{\boldsymbol{\eta}}_u^n + \widetilde{\boldsymbol{\eta}}_{u,h}^n)\|_0^2 + \sum_{M\in\mathcal{M}_h} \|\kappa_M(\nabla \boldsymbol{u}(t_n))\|_{0,M}^2 \right) \\
&\quad + \frac{1}{4} \sum_{M\in\mathcal{M}_h} \tau_M^n \|\kappa_M((\widetilde{\boldsymbol{u}}_M^n \cdot \nabla)\widetilde{\boldsymbol{e}}_{u,h}^n)\|_{0,M}^2.
\end{aligned}$$

Now we collect all the estimates and sum the resulting inequality from $n = 2$ to $m \le N$:

$$\|\widetilde{\boldsymbol{e}}_{u,h}^m\|_0^2 + \|2\boldsymbol{e}_{u,h}^m - \boldsymbol{e}_{u,h}^{m-1}\|_0^2 + \frac{4}{3}(\Delta t)^2 \|\nabla \xi_{p,h}^m\|_0^2$$

$$
\begin{aligned}
&+\sum_{n=2}^{m}\Big(\Delta t\nu\|\nabla\widetilde{\boldsymbol{e}}_{u,h}^{n}\|_0^2+\Delta t\gamma\|\nabla\cdot\widetilde{\boldsymbol{e}}_{u,h}^{n}\|_0^2+\|\delta_{tt}\boldsymbol{e}_{u,h}^{n}\|_0^2\\
&\qquad +2\Delta t\sum_{M\in\mathcal{M}_h}\tau_M^n\|\kappa_M((\widetilde{\boldsymbol{u}}_M^n\cdot\nabla)\widetilde{\boldsymbol{e}}_{u,h}^{n})\|_{0,M}^2\Big)\\
\le{}&\|\widetilde{\boldsymbol{e}}_{u,h}^{1}\|_0^2+\|2\boldsymbol{e}_{u,h}^{1}-\boldsymbol{e}_{u,h}^{0}\|_0^2+\frac{4}{3}(\Delta t)^2\|\nabla\xi_{p,h}^{1}\|_0^2\\
&+\sum_{n=2}^{m}\Bigg\{\left(\frac{1}{4}+C\frac{\Delta t}{\nu}+C\Delta t\frac{\|\widetilde{\boldsymbol{\eta}}_u^n\|_1^4+\|\widetilde{\boldsymbol{\eta}}_{u,h}^n\|_1^4}{\nu^3}\right)\|\widetilde{\boldsymbol{e}}_{u,h}^{n}\|_0^2\\
&\qquad + C\frac{\Delta t}{\nu}(\|\widetilde{\boldsymbol{\eta}}_u^n\|_1^4+\|\widetilde{\boldsymbol{\eta}}_{u,h}^n\|_1^4)\\
&\qquad + C\Delta t\max_{M\in\mathcal{M}_h}\{\tau_M^n|\widetilde{\boldsymbol{u}}_M^n|^2\}\sum_{M\in\mathcal{M}_h}\|\kappa_M(\nabla\boldsymbol{u}(t_n))\|_{0,M}^2\\
&\qquad + C\Delta t\max_{M\in\mathcal{M}_h}\{\tau_M^n|\widetilde{\boldsymbol{u}}_M^n|^2\}(\|\widetilde{\boldsymbol{\eta}}_{u,h}^n\|_1^2+\|\widetilde{\boldsymbol{\eta}}_u^n\|_1^2)\\
&\qquad + C\left(1+\frac{\Delta t}{\nu}\right)\|\widetilde{\boldsymbol{\eta}}_{u,h}^n\|_0^2+C\|\boldsymbol{\eta}_{u,h}^{n-1}\|_0^2+C\|\boldsymbol{\eta}_{u,h}^{n-2}\|_0^2+\frac{C}{\gamma}\Delta t\|\eta_{p,h}^{n-1}\|_0^2\Bigg\}.
\end{aligned}
$$

For $C\Delta t\left(1/\nu+(\|\widetilde{\boldsymbol{\eta}}_u^n\|_1^4+\|\widetilde{\boldsymbol{\eta}}_{u,h}^n\|_1^4)/\nu^3\right)<1$ we use the discrete Gronwall lemma on

$$
\left(\frac{1}{4}+C\frac{\Delta t}{\nu}+C\Delta t\frac{\|\widetilde{\boldsymbol{\eta}}_u^n\|_1^4+\|\widetilde{\boldsymbol{\eta}}_{u,h}^n\|_1^4}{\nu^3}\right)\|\widetilde{\boldsymbol{e}}_{u,h}^{n}\|_0^2
$$

and arrive at

$$
\begin{aligned}
&\|\widetilde{\boldsymbol{e}}_{u,h}^{m}\|_0^2+\|2\boldsymbol{e}_{u,h}^{m}-\boldsymbol{e}_{u,h}^{m-1}\|_0^2+\frac{4}{3}(\Delta t)^2\|\nabla\xi_{p,h}^{m}\|_0^2\\
&\quad+\sum_{n=2}^{m}\Big(\Delta t\nu\|\nabla\widetilde{\boldsymbol{e}}_{u,h}^{n}\|_0^2+\Delta t\gamma\|\nabla\cdot\widetilde{\boldsymbol{e}}_{u,h}^{n}\|_0^2+\|\delta_{tt}\boldsymbol{e}_{u,h}^{n}\|_0^2\\
&\qquad +2\Delta t\sum_{M\in\mathcal{M}_h}\tau_M^n\|\kappa_M((\widetilde{\boldsymbol{u}}_M^n\cdot\nabla)\widetilde{\boldsymbol{e}}_{u,h}^{n})\|_{0,M}^2\Big)\\
&\le C_{G,h}\left(\|\widetilde{\boldsymbol{e}}_{u,h}^{1}\|_0^2+\|2\boldsymbol{e}_{u,h}^{1}-\boldsymbol{e}_{u,h}^{0}\|_0^2+\frac{4}{3}(\Delta t)^2\|\nabla\xi_{p,h}^{1}\|_0^2\right)\\
&\quad+C_{G,h}\sum_{n=2}^{m}\Bigg\{\frac{\Delta t}{\nu}(\|\widetilde{\boldsymbol{\eta}}_u^n\|_1^4+\|\widetilde{\boldsymbol{\eta}}_{u,h}^n\|_1^4)\\
&\qquad + \Delta t\max_{M\in\mathcal{M}_h}\{\tau_M^n|\widetilde{\boldsymbol{u}}_M^n|^2\}(\sum_{M\in\mathcal{M}_h}\|\kappa_M(\nabla\boldsymbol{u}(t_n))\|_{0,M}^2)\\
&\qquad + \Delta t\max_{M\in\mathcal{M}_h}\{\tau_M^n|\widetilde{\boldsymbol{u}}_M^n|^2\}(\|\widetilde{\boldsymbol{\eta}}_{u,h}^n\|_1^2+\|\widetilde{\boldsymbol{\eta}}_u^n\|_1^2)\\
&\qquad + \left(1+\frac{\Delta t}{\nu}\right)\|\widetilde{\boldsymbol{\eta}}_{u,h}^n\|_0^2+\|\boldsymbol{\eta}_{u,h}^{n-1}\|_0^2+\|\boldsymbol{\eta}_{u,h}^{n-2}\|_0^2+\frac{\Delta t}{\gamma}\|\eta_{p,h}^{n-1}\|_0^2\Bigg\}.
\end{aligned}
$$

where $C_{G,h}$ is defined as in (2.3.17).

Due to the initial error estimates we finally obtain an estimate for the velocity terms:
For all $1 \leq m \leq N$ the discretization error due to spatial discretization can be bounded as

$$
\begin{aligned}
&\|\widetilde{\boldsymbol{e}}_{u,h}^m\|_0^2 + \|2\boldsymbol{e}_{u,h}^m - \boldsymbol{e}_{u,h}^{m-1}\|_0^2 + \frac{4}{3}(\Delta t)^2 \|\nabla \xi_{p,h}^m\|_0^2 \\
&\quad + \sum_{n=2}^{m} \Big(\Delta t \nu \|\nabla \widetilde{\boldsymbol{e}}_{u,h}^n\|_0^2 + \Delta t \gamma \|\nabla \cdot \widetilde{\boldsymbol{e}}_{u,h}^n\|_0^2 + \|\delta_{tt} \boldsymbol{e}_{u,h}^n\|_0^2 \\
&\qquad\qquad + 2\Delta t \sum_{M \in \mathcal{M}_h} \tau_M^n \|\kappa_M((\widetilde{\boldsymbol{u}}_M^n \cdot \nabla)\widetilde{\boldsymbol{e}}_{u,h}^n)\|_{0,M}^2 \Big) \\
&\leq C_{G,h} \left(\left((\nu+\gamma) h^{2k_u+2} + \gamma^{-1} h^{2k_p+4} + \Delta t^2 h^2 \left(\frac{\nu+\gamma}{\nu} + \frac{1}{\nu\gamma} \right) \right) \left(\frac{1}{\Delta t} + \frac{1}{\nu} \right) \right. \\
&\quad + \frac{C_{G,h}}{\nu^3} \left((\nu+\gamma)^2 h^{4k_u} + \gamma^{-2} h^{4k_p+4} + \Delta t^4 (\nu + \gamma + \gamma^{-1})^2 \right) \\
&\quad + C_{G,h} \max_{M \in \mathcal{M}_h} \{\tau_M^1 |\widetilde{\boldsymbol{u}}_M^1|^2\} \left(\frac{\nu+\gamma}{\nu} h^{2k_u} + \frac{h^{2k_p+2}}{\nu\gamma} + \Delta t^2 \frac{\nu+\gamma+\gamma^{-1}}{\nu} + h^{2s} \right) \\
&\quad + \frac{C_{G,h}}{\nu\gamma} \left((\nu+\gamma) h^{2k_u} + \gamma^{-1} h^{2k_p+2} + \Delta t^2 (\nu + \gamma + \gamma^{-1}) \right).
\end{aligned}
$$

□

Remark 2.3.3. Provided the intermediate solutions are sufficiently smooth in space, in the interpolation estimate for $\widetilde{\boldsymbol{\eta}}_{u,h}^n$, $\boldsymbol{\eta}_{u,h}^n$ and $\eta_{p,h}^n$ the temporal error can be skipped and the above estimate improves to give

$$
\begin{aligned}
&\|\widetilde{\boldsymbol{e}}_{u,h}^m\|_0^2 + \|2\boldsymbol{e}_{u,h}^m - \boldsymbol{e}_{u,h}^{m-1}\|_0^2 + \frac{4}{3}(\Delta t)^2 \|\nabla \xi_{p,h}^m\|_0^2 \\
&\quad + \sum_{n=2}^{m} \Big(\Delta t \nu \|\nabla \widetilde{\boldsymbol{e}}_{u,h}^n\|_0^2 + \Delta t \gamma \|\nabla \cdot \widetilde{\boldsymbol{e}}_{u,h}^n\|_0^2 + \|\delta_{tt} \boldsymbol{e}_{u,h}^n\|_0^2 \\
&\qquad\qquad + 2\Delta t \sum_{M \in \mathcal{M}_h} \tau_M^n \|\kappa_M((\widetilde{\boldsymbol{u}}_M^n \cdot \nabla)\widetilde{\boldsymbol{e}}_{u,h}^n)\|_{0,M}^2 \Big) \\
&\leq \frac{C_{G,h}}{\Delta t} \left((\nu+\gamma) h^{2k_u+2} + \gamma^{-1} h^{2k_p+4} \right) \left(1 + \frac{\Delta t}{\nu} \right) \\
&+ \frac{C_{G,h}}{\nu\gamma} \left((\nu+\gamma) h^{2k_u} + \gamma^{-1} h^{2k_p+2} \right) \\
&\quad + \frac{C_{G,h}}{\nu^3} \left((\nu+\gamma)^2 h^{4k_u} + \gamma^{-2} h^{4k_p+4} + \Delta t^4 \right) \\
&\quad + \frac{C_{G,h}}{\nu} \max_{M \in \mathcal{M}_h} \{\tau_M^1 |\widetilde{\boldsymbol{u}}_M^1|^2\} \left((\nu+\gamma) h^{2k_u} + \gamma^{-1} h^{2k_p+2} + \Delta t^2 + \nu h^{2s} \right)
\end{aligned}
\tag{2.3.24}
$$

2.4 Error Estimates for the Fully Discretized Scheme

Using the abbreviations $\boldsymbol{\zeta}_u^n := \boldsymbol{u}(t_n) - \widetilde{\boldsymbol{u}}_{ht}^n$ and $\zeta_p^n := p(t_n) - p_{ht}^n$ we combine the error estimates for the temporal (Section 2.2) and spatial discretization (cf. Lemma 2.3.2) to get the final result:

Theorem 2.4.1. *For all $1 \leq m \leq N$ the total error due to spatial discretization and discretization in time can be bounded as*

$$
\begin{aligned}
&\Delta t \sum_{n=1}^{m} \|\boldsymbol{\zeta}_u^n\|_0^2 \\
&\leq C_{G,h} \left(\left((\nu+\gamma) h^{2k_u+2} + \gamma^{-1} h^{2k_p+4} + \Delta t^2 h^2 \left(\frac{\nu+\gamma}{\nu} + \frac{1}{\nu\gamma} \right) \right) \left(\frac{1}{\Delta t} + \frac{1}{\nu} \right) \right. \\
&\quad + \frac{C_{G,h}}{\nu^3} \left((\nu+\gamma)^2 h^{4k_u} + \gamma^{-2} h^{4k_p+4} + \Delta t^4 (\nu+\gamma+\gamma^{-1})^2 \right) \\
&\quad + C_{G,h} \max_{M \in \mathcal{M}_h} \{ \tau_M^1 |\widetilde{\boldsymbol{u}}_M^1|^2 \} \left(\frac{\nu+\gamma}{\nu} h^{2k_u} + \frac{h^{2k_p+2}}{\nu\gamma} + \Delta t^2 \frac{\nu+\gamma+\gamma^{-1}}{\nu} + h^{2s} \right) \\
&\quad + \frac{C_{G,h}}{\nu\gamma} \left((\nu+\gamma) h^{2k_u} + \gamma^{-1} h^{2k_p+2} + \Delta t^2 (\nu+\gamma+\gamma^{-1}) + C(\Delta t)^4 \right) =: \chi_{u1}
\end{aligned}
$$

$$
\begin{aligned}
&\Delta t \sum_{n=1}^{m} \left(\nu \|\nabla \boldsymbol{\zeta}_u^n\|_0^2 + \gamma \|\nabla \cdot \boldsymbol{\zeta}_u^n\|_0^2 + \sum_{M \in \mathcal{M}_h} \tau_M^n \|\kappa_M((\widetilde{\boldsymbol{u}}_{ht}^n \cdot \nabla)\boldsymbol{\zeta}_u^n)\|_{0,M}^2 \right) \\
&\leq C_{G,h} \left(\left((\nu+\gamma) h^{2k_u+2} + \gamma^{-1} h^{2k_p+4} + \Delta t^2 h^2 \left(\frac{\nu+\gamma}{\nu} + \frac{1}{\nu\gamma} \right) \right) \left(\frac{1}{\Delta t} + \frac{1}{\nu} \right) \right. \\
&\quad + \frac{C_{G,h}}{\nu^3} \left((\nu+\gamma)^2 h^{4k_u} + \gamma^{-2} h^{4k_p+4} + \Delta t^4 (\nu+\gamma+\gamma^{-1})^2 \right) \\
&\quad + C_{G,h} \max_{M \in \mathcal{M}_h} \{ \tau_M^1 |\widetilde{\boldsymbol{u}}_M^1|^2 \} \left(\frac{\nu+\gamma}{\nu} h^{2k_u} + \frac{h^{2k_p+2}}{\nu\gamma} + \Delta t^2 \frac{\nu+\gamma+\gamma^{-1}}{\nu} + h^{2s} \right) \\
&\quad + \frac{C_{G,h}}{\nu\gamma} \left((\nu+\gamma) h^{2k_u} + \gamma^{-1} h^{2k_p+2} + \Delta t^2 (\nu+\gamma+\gamma^{-1}) + C(\Delta t)^2 \right) =: \chi_{u2}
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta t \sum_{n=1}^{m} \|\zeta_p^{n-1}\|_0^2 \leq C &\left(\frac{1}{(\Delta t)^2} + \|\boldsymbol{u}\|_{l^2(t_0,T;[H^2(\Omega)]^d)}^2 \right) \chi_{u1} \\
&+ \left(\nu + \gamma + \max_{1 \leq n \leq N} \max_{M \in \mathcal{M}_h} \{ \tau_M^n |\widetilde{\boldsymbol{u}}_M^n|^2 \} \right) \chi_{u2} \\
&+ C \max_{1 \leq n \leq N} \max_{M \in \mathcal{M}_h} \{ \tau_M^n |\widetilde{\boldsymbol{u}}_M^n|^2 \}^2 h^{2k_u} + C(\Delta t)^2 + C \frac{\chi_{u2}^2}{\nu^2 \Delta t}
\end{aligned}
$$

with the same Gronwall term $C_{G,h}$ as in Lemma 2.3.2.

Proof. Adding interpolation and discretization gives the estimate for all considered error norms apart from the nonlinear stabilization. All that is left is an estimate due to time discretization for this error:

$$
\begin{aligned}
&\Delta t \sum_{n=1}^{m} \sum_{M\in\mathcal{M}_h} \tau_M^n \|\kappa_M((\widetilde{\boldsymbol{u}}_{ht}^n\cdot\nabla)\boldsymbol{\zeta}_u^n)\|_{0,M}^2 \\
&\le C\Delta t \sum_{n=1}^{m} \sum_{M\in\mathcal{M}_h} \Big(\tau_M^n \|\kappa_M((\widetilde{\boldsymbol{u}}_{ht}^n\cdot\nabla)\widetilde{\boldsymbol{\xi}}_{u,h}^n)\|_{0,M}^2 + \tau_M^n \|\kappa_M((\widetilde{\boldsymbol{u}}_{ht}^n\cdot\nabla)\widetilde{\boldsymbol{\eta}}_u^n)\|_{0,M}^2\Big) \\
&\le C\Delta t \sum_{n=1}^{m} \sum_{M\in\mathcal{M}_h} \Big(\tau_M^n \|\kappa_M((\widetilde{\boldsymbol{u}}_{ht}^n\cdot\nabla)\widetilde{\boldsymbol{\xi}}_{u,h}^n)\|_{0,M}^2 + \tau_M^n |\widetilde{\boldsymbol{u}}_M^n|^2 \|\nabla\widetilde{\boldsymbol{\eta}}_u^n\|_{0,M}^2\Big) \\
&\le C\Delta t \sum_{n=1}^{m} \sum_{M\in\mathcal{M}_h} (\tau_M^n \|\kappa_M((\widetilde{\boldsymbol{u}}_{ht}^n\cdot\nabla)\widetilde{\boldsymbol{e}}_{u,h}^n)\|_{0,M}^2 \\
&\qquad\qquad + \tau_M^n |\widetilde{\boldsymbol{u}}_M^n|^2 (\|\nabla\widetilde{\boldsymbol{\eta}}_u^n\|_{0,M}^2 + \|\nabla\widetilde{\boldsymbol{\eta}}_{u,h}^n\|_{0,M}^2)).
\end{aligned}
$$

The first term is part of the left side of the discretization error estimate and the second term part of the right-hand side of the discretization error estimate. This gives the claim also for the nonlinear stabilization.

In order to obtain the estimate for the pressure error in the $L^2(\Omega)$-norm we utilize the discrete inf-sup stability of the ansatz spaces, i.e.

$$
\exists \boldsymbol{w}_h \in \boldsymbol{V}_h\colon\ \|\nabla\boldsymbol{w}_h\|_0 \le \|\zeta_p^n\|_0/\beta, \quad -(\nabla\cdot\boldsymbol{w}_h, \zeta_p^n) = \|\zeta_p^n\|_0^2. \tag{2.4.1}
$$

We test the advection-diffusion error equation with $\boldsymbol{w}_h$:

$$
\begin{aligned}
&\left(\frac{3\widetilde{\boldsymbol{\zeta}}_u^n - 4\boldsymbol{\zeta}_u^{n-1} + \boldsymbol{\zeta}_u^{n-2}}{2\Delta t}, \boldsymbol{w}_h\right) + \nu(\nabla\widetilde{\boldsymbol{\zeta}}_u^n, \nabla\boldsymbol{w}_h) + \gamma(\nabla\cdot\widetilde{\boldsymbol{\zeta}}_u^n, \nabla\cdot\boldsymbol{w}_h) \\
&= -c(\boldsymbol{u}(t_n), \boldsymbol{u}(t_n), \boldsymbol{w}_h) + c(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{w}_h) + s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{w}_h) \\
&\quad + (D_t\boldsymbol{u}(t_n) - \partial_t\boldsymbol{u}(t_n), \boldsymbol{w}_h) - (\nabla(p(t_n) - p_{ht}^{n-1}), \boldsymbol{w}_h).
\end{aligned} \tag{2.4.2}
$$

where $D_t\boldsymbol{u}(t_n) := (3\boldsymbol{u}(t_n) - 4\boldsymbol{u}(t_{n-1}) + \boldsymbol{u}(t_{n-2}))/(2\Delta t)$ and $\partial_t\boldsymbol{u}$ is the time derivative of $\boldsymbol{u}$.
Noticing $\|\boldsymbol{f}\|_{-1} \le \|\boldsymbol{f}\|_0$ we obtain

$$
\begin{aligned}
&\|\nabla\boldsymbol{w}_h\|_0 \|\zeta_p^{n-1}\|_0 \le \frac{1}{\beta}\|\zeta_p^{n-1}\|_0^2 = -(\nabla\zeta_p^{n-1}, \boldsymbol{w}_h) \\
&\le \left\|\frac{3\widetilde{\boldsymbol{\zeta}}_u^n - 4\boldsymbol{\zeta}_u^{n-1} + \boldsymbol{\zeta}_u^{n-2}}{2\Delta t}\right\|_{-1} \|\nabla\boldsymbol{w}_h\|_0 \\
&\quad + \nu\|\nabla\widetilde{\boldsymbol{\zeta}}_u^n\|_0\|\nabla\boldsymbol{w}_h\|_0 + \gamma\|\nabla\cdot\widetilde{\boldsymbol{\zeta}}_u^n\|_0\|\nabla\cdot\boldsymbol{w}_h\|_0 \\
&\quad + c(\boldsymbol{u}(t_n), \boldsymbol{u}(t_n), \boldsymbol{w}_h) - c(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{w}_h) + s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{w}_h) \\
&\quad + \|D_t\boldsymbol{u}(t_n) - \partial_t\boldsymbol{u}(t_n)\|_{-1}\|\nabla\boldsymbol{w}_h\|_0 + \|p(t_n) - p(t_{n-1})\|_0\|\nabla\cdot\boldsymbol{w}_h\|_0.
\end{aligned}
$$

We calculate for the convective terms

$$
\begin{aligned}
&c(\boldsymbol{u}(t_n),\boldsymbol{u}(t_n),\boldsymbol{w}_h) - c(\widetilde{\boldsymbol{u}}_{ht}^n,\widetilde{\boldsymbol{u}}_{ht}^n,\boldsymbol{w}_h) = c(\widetilde{\boldsymbol{\zeta}}_u^n,\boldsymbol{u}(t_n),\boldsymbol{w}_h) - c(\widetilde{\boldsymbol{u}}_{ht}^n,\widetilde{\boldsymbol{\zeta}}_u^n,\boldsymbol{w}_h)\\
&= c(\widetilde{\boldsymbol{\zeta}}_u^n,\boldsymbol{u}(t_n),\boldsymbol{w}_h) - c(\boldsymbol{u}(t_n),\widetilde{\boldsymbol{\zeta}}_u^n,\boldsymbol{w}_h) - c(\widetilde{\boldsymbol{\zeta}}_u^n,\widetilde{\boldsymbol{\zeta}}_u^n,\boldsymbol{w}_h)\\
&\le C\|\widetilde{\boldsymbol{\zeta}}_u^n\|_0\|\boldsymbol{u}(t_n)\|_2\|\boldsymbol{w}_h\|_1 + C\|\widetilde{\boldsymbol{\zeta}}_u^n\|_1^2\|\boldsymbol{w}_h\|_1
\end{aligned}
$$

and for the nonlinear stabilization

$$
\begin{aligned}
&s_h(\widetilde{\boldsymbol{u}}_{ht}^n,\widetilde{\boldsymbol{u}}_{ht}^n,\widetilde{\boldsymbol{u}}_{ht}^n,\boldsymbol{w}_h) = s_h(\widetilde{\boldsymbol{u}}_{ht}^n,\boldsymbol{u}(t_n)-\widetilde{\boldsymbol{\zeta}}_u^n,\widetilde{\boldsymbol{u}}_{ht}^n,\boldsymbol{w}_h)\\
&\le C\sum_{M\in\mathcal{M}_h}\tau_M^n|\widetilde{\boldsymbol{u}}_M^n|^2\|\kappa_M(\boldsymbol{u}(t_n))\|_{0,M}\|\|\boldsymbol{w}_h\|_{1,M}\\
&\quad + C\sum_{M\in\mathcal{M}_h}\tau_M^n\|\kappa_M((\widetilde{\boldsymbol{u}}_M^n\cdot\nabla)\widetilde{\boldsymbol{\zeta}}_u^n)\|_{0,M}|\widetilde{\boldsymbol{u}}_M^n|\|\boldsymbol{w}_h\|_{1,M}\\
&\le C\Bigg(\max_{M\in\mathcal{M}_h}\{\tau_M^n|\widetilde{\boldsymbol{u}}_M^n|^2\|\kappa_M(\boldsymbol{u}(t_n))\|_{0,M}\}\\
&\qquad + \sum_{M\in\mathcal{M}_h}\tau_M^n|\widetilde{\boldsymbol{u}}_M^n|\|\kappa_M((\widetilde{\boldsymbol{u}}_M^n\cdot\nabla)\widetilde{\boldsymbol{\zeta}}_u^n)\|_{0,M}\Bigg)\|\nabla\boldsymbol{w}_h\|_0.
\end{aligned}
$$

We combine these results and obtain due to the approximation property of κ_M and the estimates for $\|\widetilde{\boldsymbol{\zeta}}_u\|_{l^\infty(t_0,T;[L^2(\Omega)]^d)}$ and $\|\widetilde{\boldsymbol{\zeta}}_u\|_{l^2(t_0,T;LPS)}$:

$$
\begin{aligned}
&\Delta t\sum_{n=1}^N\|\zeta_p^{n-1}\|_0^2\\
&\le C\Bigg\{\frac{1}{(\Delta t)^2}\|\widetilde{\boldsymbol{\zeta}}_u\|^2_{l^\infty(t_0,T;[L^2(\Omega)]^d)} + \nu^2\|\nabla\widetilde{\boldsymbol{\zeta}}_u\|^2_{l^2(t_0,T;[L^2(\Omega)]^d)}\\
&\qquad + \gamma^2\|\nabla\cdot\widetilde{\boldsymbol{\zeta}}_u\|^2_{l^2(t_0,T;[L^2(\Omega)]^d)} + \|\widetilde{\boldsymbol{\zeta}}_u^n\|^2_{l^\infty(t_0,T;[L^2(\Omega)]^d)}\|\boldsymbol{u}\|^2_{l^2(t_0,T;[H^2(\Omega)]^d)}\\
&\qquad + \|\widetilde{\boldsymbol{\zeta}}_u^n\|^2_{l^\infty(t_0,T;[H^1(\Omega)]^d)}\|\widetilde{\boldsymbol{\zeta}}_u^n\|^2_{l^2(t_0,T;[H^1(\Omega)]^d)} + (\Delta t)^2\\
&\qquad + \max_{1\le n\le N}\max_{M\in\mathcal{M}_h}\{\tau_M^n|\widetilde{\boldsymbol{u}}_M^n|^2\}^2h^{2k_u}\|\boldsymbol{u}\|^2_{l^2(t_0,T;[W^{k_u+1,2}(\Omega)]^d)}\\
&\qquad + \max_{1\le n\le N}\max_{M\in\mathcal{M}_h}\{\tau_M^n\}\Delta t\sum_{n=1}^N\sum_{M\in\mathcal{M}_h}\tau_M^n|\widetilde{\boldsymbol{u}}_M^n|^2\|\kappa_M((\widetilde{\boldsymbol{u}}_M^n\cdot\nabla)\widetilde{\boldsymbol{\zeta}}_u^n)\|^2_{0,M}\Bigg\}\\
&\le C\left(\frac{1}{(\Delta t)^2} + \|\boldsymbol{u}\|^2_{l^2(t_0,T;[H^2(\Omega)]^d)}\right)\chi_{u1}\\
&\quad + C\left(\nu+\gamma+\max_{1\le n\le N}\max_{M\in\mathcal{M}_h}\{\tau_M^n|\widetilde{\boldsymbol{u}}_M^n|^2\}\right)\chi_{u2}\\
&\quad + C\max_{1\le n\le N}\max_{M\in\mathcal{M}_h}\{\tau_M^n|\widetilde{\boldsymbol{u}}_M^n|^2\}^2h^{2k_u}\|\boldsymbol{u}\|^2_{l^2(t_0,T;[W^{k_u+1,2}(\Omega)]^d)} + C(\Delta t)^2\\
&\quad + C\frac{\chi_{u2}^2}{\nu^2\Delta t}
\end{aligned}
$$

The requirement

$$\frac{1}{4} + C\left(\frac{\Delta t}{\nu} + \Delta t \frac{(\nu+\gamma)^2 h^{4k_u} + \gamma^{-2} h^{4k_p+4} + (\Delta t)^4 (\nu + \gamma + \gamma^{-1})^2}{\nu^5}\right) < 1$$

on the time step size Δt and the mesh width h is fulfilled if

$$\Delta t + h^{k_u} \lesssim \nu.$$

□

Corollary 2.4.2. *Provided that the stabilization parameters fulfill*

$$\gamma \in \mathcal{O}(1) \quad 0 \leq \tau_M^n \leq \frac{C}{|\widetilde{\boldsymbol{u}}_M^n|^2} \tag{2.4.3}$$

and the mesh size h and the time step size Δt are chosen according to

$$\Delta t, h^{k_u} \lesssim \nu \tag{2.4.4}$$

the above estimates state (omitting the remaining ν-dependencies on the right hand side)

$$\begin{aligned}
&\Delta t \sum_{n=1}^{m} \|\boldsymbol{\zeta}_u^n\|_0^2 \\
&\quad + \Delta t \sum_{n=1}^{m} \left(\nu \|\nabla \boldsymbol{\zeta}_u^n\|_0^2 + \gamma \|\nabla \cdot \boldsymbol{\zeta}_u^n\|_0^2 + \sum_{M \in \mathcal{M}_h} \tau_M^n \|\kappa_M((\widetilde{\boldsymbol{u}}_{ht}^n \cdot \nabla)\boldsymbol{\zeta}_u^n)\|_{0,M}^2 \right) \\
&\leq C_{G,h} \left(\frac{1}{\Delta t} + \frac{1}{h^2} \right) (h^{2k_p+4} + h^{2k_u+2} + h^2 (\Delta t)^2) \\
&\textit{and} \\
&\Delta t \sum_{n=1}^{m} \|\zeta_p^{n-1}\|_0^2 \leq C_{G,h} \left(\frac{1}{(\Delta t)^3} + \frac{1}{h^2 (\Delta t)^2} \right) (h^{2k_p+4} + h^{2k_u+2} + h^2 (\Delta t)^2).
\end{aligned} \tag{2.4.5}$$

Remark 2.4.3. Our ansatz spaces typically satisfy $k_u = k_p + 1$ and therefore an equilibration of error terms for the velocity suggests $h^2 \lesssim \Delta t \lesssim 1$. Then Corollary 2.4.5 tells us that our scheme converges as $\min\{h^{k_u}, h^2\}$ with respect to the velocity errors. For the LPS error this estimate is the best that can be expected. However, for the $L^2(\Omega)$ error this result is suboptimal. We would expect a behavior according to h^{2k_u+2}. Finally, for the pressure the estimates just tells us that the error is bounded.
If we assume more regularity for our time discretized quantities, i.e.

$$\boldsymbol{u}_t \in L^\infty(t_0, T; [H^{k_u+1}(\Omega)]^d) \tag{2.4.6}$$

the results due to Theorem 2.4.1 (using the assumptions in Corollary 2.4.5) improve to give

$$\Delta t \sum_{n=1}^{m} \|\boldsymbol{\zeta}_u^n\|_0^2 \leq C_{G,h} h^{2k_p+2} \Delta t + C_{G,h} \frac{h^{2k_u+2}}{\Delta t} + C_{G,h} h^{2k_u} + C_{G,t} (\Delta t)^4$$

$$\Delta t \sum_{n=1}^{m} \left(\nu \|\nabla \boldsymbol{\zeta}_u^n\|_0^2 + \gamma \|\nabla \cdot \boldsymbol{\zeta}_u^n\|_0^2 + \sum_{M \in \mathcal{M}_h} \tau_M^n \|\kappa_M((\widetilde{\boldsymbol{u}}_{ht}^n \cdot \nabla)\boldsymbol{\zeta}_u^n)\|_{0,M}^2 \right)$$
$$\leq C_{G,h} h^{2k_p+2} \Delta t + C_{G,h} \frac{h^{2k_u+2}}{\Delta t} + C_{G,t} (\Delta t)^2 + C_{G,h} h^{2k_u} \tag{2.4.7}$$

and

$$\Delta t \sum_{n=1}^{m} \|\zeta_p^{n-1}\|_0^2$$
$$\leq C_{G,h} \frac{h^{2k_p+2}}{\Delta t} + C_{G,h} \frac{h^{2k_u+2}}{(\Delta t)^3} + C_{G,h} \frac{h^{2k_u}}{(\Delta t)^2} + h^{2k_u} + C(\Delta t)^2.$$

Again we loose one order of convergence with respect to the $L^2(\Omega)$ error of the velocity no matter how Δt is chosen. But now the order of convergence with respect to Δt is bounded by the temporal approximation for all considered quantities and norms. Equilibrating Δt and h with respect to the LPS norm again gives the choice $h^2 \lesssim \Delta t \lesssim 1$ and the error converges as before like $h^{k_u} + \Delta t$. For the pressure the choice $\Delta t \sim h$ gives convergence according to $h^{k_u - 1}$. As for the $L^2(\Omega)$ velocity error we loose an order of convergence with respect to h.
The reason for the suboptimal errors with respect to space is that the bound for the velocity energy error is the same as for the LPS error. In order to improve this we would need a superconvergent discretization error estimate and hence an interpolation operator that is better suited to our problem. The authors will perform such an approach in future work.

Remark 2.4.4. For the discretization in time we assumed that the velocity can be bounded in the $L^\infty(t_0, T; [H^2(\Omega)]^d)$ norm. Therefore we were able to bound the error due to time discretization by a Gronwall term that scales like $C_{G,t} \sim \exp\left(\frac{T}{1 - \Delta t / \nu}\right)$. In the spatial discretization we did not use such a bound, but only assumed the discretized velocity $\widetilde{\boldsymbol{u}}_t$ to be in $L^\infty(t_0, T; [H^1(\Omega)]^d)$. As a consequence the Gronwall term due to spatial discretization behaves like $C_{G,h} \sim \exp\left(\frac{T}{1-K}\right)$ with $K \lesssim \Delta t \frac{h^{4k_u} \nu^2 + (\Delta t)^4}{\nu^5}$. If we would have a bound for $\boldsymbol{u}_t$ in the $L^\infty(t_0, T; [H^1(\Omega)]^d)$ norm, we could have proven a bound for $\widetilde{\boldsymbol{u}}_t$ in the $L^\infty(t_0, T; [H^2(\Omega)]^d)$ norm and hence improve the behavior in the Gronwall term

$C_{G,h}$with respect to ν:

$$\begin{aligned}
&c(\widetilde{\boldsymbol{u}}_t, \widetilde{\boldsymbol{u}}_t, \widetilde{\boldsymbol{e}}^n_{u,h}) - c(\widetilde{\boldsymbol{u}}^n_{ht}, \widetilde{\boldsymbol{u}}^n_{ht}, \widetilde{\boldsymbol{e}}^n_{u,h})\\
&= c(\widetilde{\boldsymbol{\eta}}^n_u + \widetilde{\boldsymbol{e}}^n_{u,h}, \widetilde{\boldsymbol{u}}_t, \widetilde{\boldsymbol{e}}^n_{u,h}) + c(\widetilde{\boldsymbol{u}}^n_{ht}, \widetilde{\boldsymbol{\eta}}^n_u + \widetilde{\boldsymbol{e}}^n_{u,h}, \widetilde{\boldsymbol{e}}^n_{u,h})\\
&= c(\widetilde{\boldsymbol{\eta}}^n_u + \widetilde{\boldsymbol{e}}^n_{u,h}, \widetilde{\boldsymbol{u}}_t, \widetilde{\boldsymbol{e}}^n_{u,h}) + c(\widetilde{\boldsymbol{u}}^n_{ht}, \widetilde{\boldsymbol{\eta}}^n_u, \widetilde{\boldsymbol{e}}^n_{u,h})\\
&= c(\widetilde{\boldsymbol{e}}^n_{u,h}, \widetilde{\boldsymbol{u}}_t, \widetilde{\boldsymbol{e}}^n_{u,h}) + c(\widetilde{\boldsymbol{\eta}}^n_u, \widetilde{\boldsymbol{u}}_t, \widetilde{\boldsymbol{e}}^n_{u,h}) + c(\widetilde{\boldsymbol{u}}_t, \widetilde{\boldsymbol{\eta}}^n_u, \widetilde{\boldsymbol{e}}^n_{u,h})\\
&\quad - c(\widetilde{\boldsymbol{\eta}}^n_u, \widetilde{\boldsymbol{\eta}}^n_u, \widetilde{\boldsymbol{e}}^n_{u,h}) - c(\widetilde{\boldsymbol{e}}^n_{u,h}, \widetilde{\boldsymbol{\eta}}^n_u, \widetilde{\boldsymbol{e}}^n_{u,h})
\end{aligned} \tag{2.4.8}$$

$$\begin{aligned}
&c(\widetilde{\boldsymbol{e}}^n_{u,h}, \widetilde{\boldsymbol{u}}_t, \widetilde{\boldsymbol{e}}^n_{u,h}) \leq C\|\widetilde{\boldsymbol{e}}^n_{u,h}\|_0 \|\widetilde{\boldsymbol{u}}_t\|_2 \|\widetilde{\boldsymbol{e}}^n_{u,h}\|_1\\
&\quad \leq \frac{\nu}{32}\|\nabla \widetilde{\boldsymbol{e}}^n_{u,h}\|_0^2 + \frac{C}{\nu}\|\widetilde{\boldsymbol{e}}^n_{u,h}\|_0^2\\
&c(\widetilde{\boldsymbol{u}}_t, \widetilde{\boldsymbol{\eta}}^n_u, \widetilde{\boldsymbol{e}}^n_{u,h}) + c(\widetilde{\boldsymbol{\eta}}^n_u, \widetilde{\boldsymbol{u}}_t, \widetilde{\boldsymbol{e}}^n_{u,h})\\
&\quad \leq C\|\widetilde{\boldsymbol{u}}(t_n)\|_2 \|\widetilde{\boldsymbol{\eta}}^n_u\|_0 \|\widetilde{\boldsymbol{e}}^n_{u,h}\|_1 \leq \frac{\nu}{32}\|\nabla \widetilde{\boldsymbol{e}}^n_{u,h}\|_0^2 + \frac{C}{\nu}\|\widetilde{\boldsymbol{\eta}}^n_u\|_0^2\\
&c(\widetilde{\boldsymbol{\eta}}^n_u, \widetilde{\boldsymbol{\eta}}^n_u, \widetilde{\boldsymbol{e}}^n_{u,h}) \leq C\|\widetilde{\boldsymbol{\eta}}^n_u\|_1^2 \|\widetilde{\boldsymbol{e}}^n_{u,h}\|_1 \leq \frac{\nu}{32}\|\nabla \widetilde{\boldsymbol{e}}^n_{u,h}\|_0^2 + \frac{C}{\nu}\|\widetilde{\boldsymbol{\eta}}^n_u\|_1^4\\
&c(\widetilde{\boldsymbol{e}}^n_{u,h}, \widetilde{\boldsymbol{\eta}}^n_u, \widetilde{\boldsymbol{e}}^n_{u,h}) \leq C\|\widetilde{\boldsymbol{\eta}}^n_u\|_1 \|\widetilde{\boldsymbol{e}}^n_{u,h}\|_1^2
\end{aligned} \tag{2.4.9}$$

and therefore

$$\begin{aligned}
&c(\widetilde{\boldsymbol{u}}_t, \widetilde{\boldsymbol{u}}_t, \widetilde{\boldsymbol{e}}^n_{u,h}) - c(\widetilde{\boldsymbol{u}}^n_{ht}, \widetilde{\boldsymbol{u}}^n_{ht}, \widetilde{\boldsymbol{e}}^n_{u,h})\\
&\leq C\|\nabla \widetilde{\boldsymbol{e}}^n_{u,h}\|_0^2 \left(\nu + \|\widetilde{\boldsymbol{\eta}}^n_u\|_1\right) + \frac{C}{\nu}\|\widetilde{\boldsymbol{e}}^n_{u,h}\|_0^2 + \frac{C}{\nu}\|\widetilde{\boldsymbol{\eta}}^n_{u,h}\|_0^2 + \frac{C}{\nu}\|\widetilde{\boldsymbol{\eta}}^n_{u,h}\|_1^4.
\end{aligned}$$

Provided $\|\widetilde{\boldsymbol{\eta}}^n_u\|_1 \leq C\nu$ this gives a Gronwall term according to $C_{G,h} \sim \exp\left(\frac{T}{1-\Delta t/\nu}\right)$.

Chapter 3

Spatial-Temporal Discretization

In this approach we first discretize in space and afterwards in time. This means that we can bound our errors according to the strategy

$$\|\boldsymbol{U} - \boldsymbol{U}_{ht}\| \leq \|\boldsymbol{U} - \boldsymbol{U}_h\| + \|\boldsymbol{U}_h - \boldsymbol{U}_{ht}\|. \tag{3.0.1}$$

The errors resulting from $\boldsymbol{u} - \boldsymbol{u}_h$ are considered in Section 3.1. The estimates for $\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}_{ht}$ are presented in Section 3.3. In Section 3.4 we will derive an error bound on $\|p - p_{ht}\|_{l^2(t_0,T;L^2(\Omega))}$ via the inf-sup stability of the ansatz spaces and combine all the estimates.

The assumptions that we impose for this approach are given by

Assumption 3.0.5. The spatially discretized quantities fulfill the regularity requirement

$$\begin{aligned} &\boldsymbol{u}_h \in W^{1,\infty}(t_0,T;[L^2(\Omega)]^d) \cap W^{l,2}(t_0,T;[H^1(\Omega)]^d) \\ &p_h \in W^{2,\infty}(t_0,T;H^1(\Omega)) \end{aligned} \tag{3.0.2}$$

with $l \in \{1, 2\}$. Further it holds

$$\begin{aligned} \|\boldsymbol{u}(t)\|_2 + \|\partial_t \boldsymbol{u}(t)\|_0 + \|p(t)\|_1 \leq C \quad \forall t \in [0,T] \\ \boldsymbol{u} \in L^\infty(t_0,T;[W^{k_u+1,2}(\Omega)]^d) \end{aligned} \tag{3.0.3}$$

and for any $1 \leq n \leq N$

$$\widetilde{\boldsymbol{u}}_{ht}^n \in \boldsymbol{V} \cap [W^{l,2}(\Omega)]^d. \tag{3.0.4}$$

For the semi-discrete error we also need:

$$\begin{gathered} \boldsymbol{u} \in L^\infty(t_0,T;[W^{1,\infty}(\Omega)]^d) \cap L^\infty(t_0,T;[W^{k_u+1,2}(\Omega)]^d), \\ \partial_t \boldsymbol{u} \in L^2(t_0,T;[W^{k_u,2}(\Omega)]^d),\ p \in L^2(t_0,T;W^{k_p+1,2}(\Omega)). \end{gathered}$$

3.1 Quasi-optimal Spatial Error Estimates

Assuming that the assumptions 1.1.1, 1.1.2, 1.1.3 and 1.1.4 hold true we can state here the main result from our previous paper [8].

Theorem 3.1.1. *Assume that* $\boldsymbol{u}_h(t_0) = j_u \boldsymbol{u}_0$*. If* $\boldsymbol{u} \in L^\infty(t_0, T; [W^{1,\infty}(\Omega)]^d)$*, then we obtain for the discrete velocity approximation* $\boldsymbol{e}_h = \boldsymbol{u}_h - j_u \boldsymbol{u}$ *of the LPS-method (1.1.11):*

$$\begin{aligned}
&\|\boldsymbol{e}_h\|^2_{L^\infty(0,t);[L^2(\Omega)]^d} + \int_{t_0}^T |||\boldsymbol{e}_h(\tau)|||^2_{LPS}\, d\tau \\
&\leq C \sum_M \int_{t_0}^T e^{C_G(\boldsymbol{u})(t-\tau)} \Big[(\nu + \tau_M |\boldsymbol{u}_M|^2 + \gamma d) \|\nabla \boldsymbol{\eta}_u(\tau)\|^2_{0,M} \\
&\quad + (1 + \nu Re_M^2) h^{-2} \|\boldsymbol{\eta}_u(\tau)\|^2_{0,M} + \|\partial_t \boldsymbol{\eta}_u(\tau)\|^2_{0,M} \\
&\quad + \tau_M |\boldsymbol{u}_M|^2 \|\kappa_M(\nabla \boldsymbol{u})(\tau)\|^2_{0,M} + \min\left(\frac{d}{\nu}; \frac{1}{\gamma}\right) \|\eta_p(\tau)\|^2_{0,M} \Big]\, d\tau
\end{aligned} \tag{3.1.1}$$

with $(\boldsymbol{\eta}_u, \eta_p) = (\boldsymbol{u} - j_u \boldsymbol{u}, p - j_p p)$*, the local Reynolds number* $Re_M := \frac{h\|\boldsymbol{u}_h\|_{L^\infty(M)}}{\nu}$*, and the Gronwall constant*

$$C_G(\boldsymbol{u}) = 1 + C|\boldsymbol{u}|_{L^\infty(t_0,T;[W^{1,\infty}(\Omega)]^d)} + Ch\|\boldsymbol{u}\|^2_{L^\infty(t_0,T;[W^{1,\infty}(\Omega)]^d)} \tag{3.1.2}$$

where $h := \max_M h$*.*

Using this estimate we can deduce quasi-optimal error estimates.

Corollary 3.1.2. *Assume a smooth solution of the time-dependent Navier-Stokes-problem according to*

$$\begin{aligned}
\boldsymbol{u} &\in L^\infty(t_0, T; [W^{1,\infty}(\Omega)]^d) \cap L^2(t_0, T; [W^{k_u+1,2}(\Omega)]^d), \\
\partial_t \boldsymbol{u} &\in L^2(t_0, T; [W^{k_u,2}(\Omega)]^d),\ p \in L^2(t_0, T; W^{k_p+1,2}(\Omega)).
\end{aligned}$$

and let $\boldsymbol{u}_h(t_0) = j_u \boldsymbol{u}_0$*. Then we obtain for* $0 \leq t \leq T$ *the semi-discrete a-priori estimate for the approximation* $\boldsymbol{e}_h = \boldsymbol{u}_h - j_u \boldsymbol{u}$ *of the LPS-method (1.1.11):*

$$\begin{aligned}
&\|\boldsymbol{e}_h\|^2_{L^\infty(0,t);[L^2(\Omega)]^d} + \int_{t_0}^T |||\boldsymbol{e}_h(\tau)|||^2_{LPS} d\tau \\
&\leq C \sum_M h^{2k_u} \int_{t_0}^T e^{C_G(\boldsymbol{u})(t-\tau)} \Big(\Big(1 + \nu Re_M^2 + \tau_M |\boldsymbol{u}_M|^2 + d\gamma\Big) |\boldsymbol{u}(\tau)|^2_{W^{k_u+1,2}(\omega_M)} \\
&\quad + \tau_M |\boldsymbol{u}_M|^2 h^{2(s-k_u)} |\boldsymbol{u}(\tau)|^2_{W^{s+1,2}(\omega_M)} + |\partial_t \boldsymbol{u}(\tau)|^2_{W^{k_u,2}(\omega_M)} \\
&\quad + h^{2(k_p+1-k_u)} \min\left(\frac{d}{\nu}, \frac{1}{\gamma}\right) |p(\tau)|^2_{W^{k_p+1,2}(\omega_M)} \Big) d\tau
\end{aligned}$$

with the mesh Reynolds number $Re_M := \frac{h\|\boldsymbol{u}_h\|_{L^\infty(M)}}{\nu}$ *and* $s \in \{0, \cdots, k_u\}$*.*

Remark 3.1.3. The error estimate (3.1.2) does not blow up if

$$Re_M = \frac{h\|\boldsymbol{u}_h\|_{L^\infty(M)}}{\nu} \le \frac{1}{\sqrt{\nu}} \tag{3.1.3}$$

which gives a restriction on the local mesh width h. Thus we obtain a method of order k provided that $Re_M \le C/\sqrt{\nu}$. This condition is less restrictive than the usual condition $Pe_M := h\|\boldsymbol{b}\|_{L^\infty(M)}/\nu \le C$ for the Galerkin method applied to advection-diffusion problems where $\boldsymbol{b}$ is a stationary velocity field. An alternative stability estimate is given in [12] for the stationary Oseen problem which requires the more restrictive condition $Re_\Omega := \frac{\|\boldsymbol{b}\|_{L^\infty(\Omega)} C_P}{\nu} \le \frac{1}{\sqrt{\nu}}$. □

The approach of this subsection is applicable to almost all LPS-variants. We summarize possible variants of the triples $\boldsymbol{V}_h/Q_h/D_M$ with $t \in \{0, \dots, k-1\}$:

- One-level methods:
  $\mathbb{P}_k/\mathbb{P}_{k-1}/\mathbb{P}_t$, $\mathbb{Q}_k/\mathbb{Q}_{k-1}/\mathbb{Q}_t$, $\mathbb{P}_k^+/\mathbb{P}_{-(k-1)}/\mathbb{P}_t$, $\mathbb{Q}_k/\mathbb{P}_{-(k-1)}/\mathbb{P}_t$
- Two-level methods:
  $\mathbb{P}_k/\mathbb{P}_{k-1}/\mathbb{P}_t$, $\mathbb{Q}_k/\mathbb{Q}_{k-1}/\mathbb{Q}_t$, $\mathbb{P}_k^+/\mathbb{P}_{-(k-1)}/\mathbb{P}_t$, $\mathbb{Q}_k/\mathbb{P}_{-(k-1)}/\mathbb{P}_t$.

3.2 Approximating the Spatial Results with respect to Time

In order to combine the error estimates for the spatial discretization with those obtained from the discretization in time we need to derive an interpolation result. This relies on the following well-known extension of the Bramble-Hilbert theorem:

Theorem 3.2.1. *Let X be a separable Hilbert space. Then there exists $C > 0$ such that for any bounded interval $(a,b) \subset \mathbb{R}$ and for any $f \in H^m((a,b);X)$ there exists a polynomial $q \in \mathbb{P}^{m-1}(X)$ satisfying $q(a) = f(a)$ and $q(b) = f(b)$ and*

$$\|f - q\|_{H^k((a,b);X)} \le C(b-a)^{m-k}|f|_{H^m((a,b);X)} \qquad \forall k \le m. \tag{3.2.1}$$

Proof. Follow the presentation in [13, Chapter 4]. The basic steps are:

- Consider the averaged Taylor polynomial

  $$Q^m f(x) := \int_a^b \sum_{k<m} \frac{1}{k!} D^k f(y)(x-y)^k \phi(y)\, dy$$

  where $\phi \in C_0^\infty(\mathbb{R})$ has the properties $\operatorname{supp}\phi = [a,b]$ and $\int_a^b \phi(x)\, dx = 1$.

- For the remainder $R^m f := f - Q^m f$ find a representation of form

  $$R^m f(x) = m \int_a^b \alpha_k(x,z) D^k f(z)\, dz,$$

  where $z = x + s(y-x)$, $\alpha_k(x,z) = (x-z)^k/(k!)\alpha(x,z)$ and

  $$|\alpha(x,z)| \le C|z-x|^{-1}.$$

- Estimate $R^m f$ according to

  $$|R^m f|_{H^m((a,b);X)} \le C(b-a)^{l-m}|f|_{H^l((a,b);X)}$$

  in case of $m \le l$.

□

Furthermore, we need the following Lemma on the equivalence on the discrete l^2 and the continuous L^2 norm for finite element functions:

Lemma 3.2.2. *Consider the set of points in time $M_T = \{t_0, \ldots, t_N = T\}$ where we assume a constant time step size $\Delta t = (T - t_0)/N$ and let X be a Banach space. Then there exist constants c, C such that the estimate*

$$c\Delta t \sum_{i=0}^{N} \|f(t_i)\|_X^2 \le \|f\|_{L^2(t_0,T;X)}^2 \le C\Delta t \sum_{i=0}^{N} \|f(t_i)\|_X^2 \tag{3.2.2}$$

holds true for all functions $f : [t_0, T] \to X$ that are piecewise linear with respect to. to M_T.

Proof. By expanding one observes

$$\begin{aligned}
&\int_{t_{i-1}}^{t_i} \|f(t)\|_X^2\, dt = \Delta t \frac{\Delta t}{6}\left(\|f(t_{i-1})\|_X^2 + 4\|f\left(\frac{t_{i-1}+t_i}{2}\right)\|_X^2 + \|f(t_i)\|_X^2\right) \\
&= \frac{\Delta t}{6}(\|f(t_{i-1})\|_X^2 + (\|f(t_{i-1})\|_X^2 + (f(t_{i-1}), f(t_i))_X + \|f(t_i)\|_X^2) + \|f(t_i)\|_X^2) \\
&= \frac{\Delta t}{3}(\|f(t_{i-1})\|_X^2 + (f(t_{i-1}), f(t_i)_X + \|f(t_i)\|_X^2))
\end{aligned}$$

and therefore

$$\frac{\Delta t}{6}(\|f(t_{i-1})\|_X^2 + \|f(t_i)\|_X^2) \le \int_{t_{i-1}}^{t_i} \|f(t)\|_X^2\, dt \le \frac{\Delta t}{2}(\|f(t_{i-1})\|_X^2 + \|f(t_i)\|_X^2).$$

Due to

$$\int_{t_0}^{T} \|f(t)\|_X^2\, dt = \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \|f(t)\|_X^2\, dt = \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \|f(t)\|_X^2\, dt$$

this yields

$$\sum_{i=1}^{N} \frac{\Delta t}{6}(\|f(t_{i-1})\|_X^2 + \|f(t_i)\|_X^2) \le \int_{t_0}^{T} \|f(t)\|_X^2 \, dt$$
$$\int_{t_0}^{T} \|f(t)\|_X^2 \, dt \le \sum_{i=1}^{N} \frac{\Delta t}{2}(\|f(t_{i-1})\|_X^2 + \|f(t_i)\|_X^2)$$

and finally

$$\frac{\Delta t}{6} \sum_{i=0}^{N} \|f(t_n)\|_X^2 \le \int_{t_0}^{T} \|f(t)\|_X^2 \, dt \le \Delta t \sum_{i=0}^{N} \|f(t_n)\|_X^2.$$

□

Now we can reformulate our spatial results with respect to the time discretization:

Corollary 3.2.3. *The results of the spatial discretization with respect to the discrete norm may be written as*

$$\begin{aligned} \|\boldsymbol{\xi}_{u,h}\|^2_{l^\infty(t_0,T;[L^2(\Omega)]^d)} &\le \|\boldsymbol{\xi}_{u,h}\|^2_{L^\infty(t_0,T;[L^2(\Omega)]^d)} \le Ch^{2k_u} \\ \|\boldsymbol{\xi}_{u,h}\|^2_{l^2(t_0,T;LPS)} &\le C\|\boldsymbol{\xi}_{u,h}\|^2_{L^2(t_0,T;LPS)} + C(\Delta t)^{2l} \\ &\le C(h^{2k_u} + (\Delta t)^{2l}) \end{aligned} \tag{3.2.3}$$

where $\boldsymbol{\xi}_{u,h} := \boldsymbol{u} - \boldsymbol{u}_h$ *provided* $\boldsymbol{\xi}_{u,h} \in H^l(t_0, T; LPS)$, $l \in \{1, 2\}$.

Proof. Due to definitions of the norms we get for the first claim

$$\|\boldsymbol{\xi}_{u,h}\|^2_{l^2(t_0,T;[L^2(\Omega)]^d)} \le \|\boldsymbol{\xi}_{u,h}\|^2_{l^\infty(t_0,T;[L^2(\Omega)]^d)} \quad \le \|\boldsymbol{\xi}_{u,h}\|^2_{L^\infty(t_0,T;[L^2(\Omega)]^d)}.$$

For the second statement choose the partitioning $\mathcal{M}_h := ([t_0 + (i-1)\Delta t, t_0 + i\Delta t])_{i=1,\dots,(T-t_0)/\Delta t}$ in the previous lemma and a piecewise linear nodal basis. For $\boldsymbol{v}_h \in L^2(t_0, T; LPS)$ let $\boldsymbol{v}_{ht} := \boldsymbol{I}\boldsymbol{v}_h := \sum_i \boldsymbol{v}_h(t_i)\phi_i$ be a finite element approximation.
In conjunction with Sobolev's Inequality [13, Lemma 4.3.4]

$$\|u\|_{L^\infty(a,b,X)} \le C\|u\|_{H^l(a,b;X)}$$

the above generalized Bramble-Hilbert theorem (3.2.1) gives

$$\begin{aligned} &\|\boldsymbol{v}_h - \boldsymbol{v}_{ht}\|_{L^2(t_0,T;X)} \\ &\le \|\boldsymbol{v}_h - Q^l\boldsymbol{v}_h\|_{L^2(t_0,T;X)} + \|Q^l\boldsymbol{v}_h - \boldsymbol{v}_{ht}\|_{L^2(t_0,T;X)} \\ &\le \sum_{i=1}^{N} \left(\|\boldsymbol{v}_h - Q^l\boldsymbol{v}_h\|_{L^2(t_{i-1},t_i;X)} + \|I(Q^l\boldsymbol{v}_h - \boldsymbol{v}_h)\|_{L^2(t_{i-1},t_i;X)}\right) \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{N} \left(\|\boldsymbol{v}_h - Q^l \boldsymbol{v}_h\|_{L^2(t_{i-1},t_i;X)} + \Delta t \|Q^l \boldsymbol{v}_h - \boldsymbol{v}_h\|_{L^\infty(t_{i-1},t_i;X)} \right) \\
&\leq \sum_{i=1}^{N} (1+\Delta t) \|\boldsymbol{v}_h - Q^l \boldsymbol{v}_h\|_{L^2(t_{i-1},t_i;X)} \leq \sum_{i=1}^{N} C(\Delta t)^l \|\boldsymbol{v}_h\|_{H^l(t_{i-1},t_i;X)} \\
&\leq C(\Delta t)^l \|\boldsymbol{v}_h\|_{H^l(t_0,T;X)}.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
c\|\boldsymbol{v}_{ht}\|^2_{l^2(t_0,T;LPS)} &\leq \|\boldsymbol{v}_{ht}\|^2_{L^2(t_0,T;LPS)} \\
&\leq C\|\boldsymbol{v}_h\|^2_{L^2(t_0,T;LPS)} + C\|\boldsymbol{v}_{ht} - \boldsymbol{v}_h\|^2_{L^2(t_0,T;LPS)} \\
&\leq C\|\boldsymbol{v}_h\|^2_{L^2(t_0,T;LPS)} + C(\Delta t)^{2l}.
\end{aligned}
$$

This means that we can estimate $\boldsymbol{\xi}_{u,h}$ by

$$
\begin{aligned}
c\|\boldsymbol{\xi}_{u,h}\|^2_{l^2(t_0,T;LPS)} &\leq \|\boldsymbol{\xi}_{u,ht}\|^2_{l^2(t_0,T;LPS)} + \|\boldsymbol{\xi}_{u,ht} - \boldsymbol{\xi}_{u,h}\|^2_{l^2(t_0,T;LPS)} \\
&\leq C\|\boldsymbol{\xi}_{u,ht}\|^2_{L^2(t_0,T;LPS)} + C(\Delta t)^{2l}.
\end{aligned}
$$

□

In the following estimates we need to bound $\boldsymbol{u}_h$ in the $l^\infty(H^1(\Omega))$ norm by using the following result.

Lemma 3.2.4. *$\boldsymbol{u}_h$ can be bounded in the $l^\infty(H^1(\Omega))$-norm provided $\boldsymbol{\xi}_{u,h} \in H^l(t_0,T;LPS)$, $l \in \{1,2\}$ and, provided $h^{2k_u} \leq \nu\Delta t$, it holds*

$$
\|\boldsymbol{u}_h\|^2_{l^\infty(H^1(\Omega))} \leq C, \quad \|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{l^\infty(H^1(\Omega))} \leq \frac{C}{\Delta t}\left(\frac{h^{2k_u}}{\nu} + (\Delta t)^{2l}\right).
$$

Proof. We estimate using Corollary 3.2.3

$$
\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{l^\infty(H^1(\Omega))} &\leq \frac{C}{\Delta t}\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{l^2(t_0,T;[H^1(\Omega)]^d)} \\
&\leq \frac{C}{\Delta t}\left(\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{L^2(t_0,T;[H^1(\Omega)]^d)} + (\Delta t)^{2l}\right) \\
&\leq C\frac{h^{2k_u}}{\nu\Delta t} + C(\Delta t)^{2l-1} \leq \frac{C}{\Delta t}\left(\frac{h^{2k_u}}{\nu} + (\Delta t)^{2l}\right).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|\boldsymbol{u}_h\|^2_{l^\infty(H^1(\Omega))} &\leq C\|\boldsymbol{u}\|^2_{l^\infty(H^1(\Omega))} + \frac{C}{\Delta t}\|\boldsymbol{u} - \boldsymbol{u}_h\|_{l^2(t_0,T;[H^1(\Omega)]^d)} \\
&\leq C\|\boldsymbol{u}\|^2_{l^\infty(H^1(\Omega))} + C\frac{h^{2k_u}}{\Delta t} + C(\Delta t)^{2l-1} \\
&\leq C + \frac{C}{\Delta t}\left(\frac{h^{2k_u}}{\nu} + (\Delta t)^{2l}\right) \leq C
\end{aligned}
$$

provided $h^{2k_u} \leq \nu\Delta t$. □

Assumption 3.2.5. We assume that the spatially discretized pressure solution solution is sufficiently smooth, i.e. $p_h \in H^2(t_0,T;L^2(\Omega))$.

3.3 Time Discretization

We first want to consider the convective term and its stabilization explicitly. Afterwards we derive bound for the additional error due to the nonlinearity. This means that we can estimate the last term in (3.0.1) as

$$\|\boldsymbol{U}_h - \boldsymbol{U}_{ht}\| \leq \|\boldsymbol{U}_h - \boldsymbol{W}_{ht}\| + \|\boldsymbol{W}_{ht} - \boldsymbol{U}_{ht}\| \tag{3.3.1}$$

where $\boldsymbol{W}_{ht} = (\boldsymbol{w}_{ht}, r_{ht})$ solves the auxiliary problem

Find $\widetilde{\boldsymbol{w}}_{ht}^n \in \boldsymbol{V}_h, \boldsymbol{w}_{ht}^n \in \boldsymbol{Y}_h$ and $r_{ht}^n \in Q_h$ such that

$$\begin{aligned}
&\left(\frac{3\widetilde{\boldsymbol{w}}_{ht}^n - 4\boldsymbol{w}_{ht}^{n-1} + \boldsymbol{w}_{ht}^{n-2}}{2\Delta t}, \boldsymbol{v}_h\right) + \nu(\nabla \widetilde{\boldsymbol{u}}_{ht}^n, \nabla \boldsymbol{v}_h) + t_h(\widetilde{\boldsymbol{u}}_{ht}^n; \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) \\
&= (\boldsymbol{f}^n, \boldsymbol{v}_h) - (\nabla r_{ht}^{n-1}, \boldsymbol{v}_h) - c(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) - s_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) \\
&\widetilde{\boldsymbol{w}}_{ht}^n|_{\partial\Omega} = 0
\end{aligned} \tag{3.3.2}$$

$$\begin{aligned}
\left(\frac{3\boldsymbol{w}_{ht}^n - 3\widetilde{\boldsymbol{w}}_{ht}^n}{2\Delta t} + \nabla(r_{ht}^n - r_{ht}^{n-1}), \boldsymbol{y}_h\right) &= 0 \\
(\nabla \cdot \boldsymbol{w}_{ht}^n, q_h) &= 0 \\
\boldsymbol{w}_{ht}^n|_{\partial\Omega} &= 0
\end{aligned} \tag{3.3.3}$$

for all $\boldsymbol{v}_h \in \boldsymbol{V}_h, \boldsymbol{y}_h \in \boldsymbol{Y}_h$ and $q_h \in Q_h$.

3.3.1 Notation

In order to abbreviate the representation of errors, we introduce some notations.

Definition 3.3.1. For the time discretized problem denote

$$\boldsymbol{\xi}_u^n := \boldsymbol{u}_h(t_n) - \boldsymbol{u}_{ht}^n \quad \widetilde{\boldsymbol{\xi}}_u^n := \boldsymbol{u}_h(t_n) - \widetilde{\boldsymbol{u}}_{ht}^n \qquad \xi_p^n := p_h(t_n) - p_{ht}^n$$

For the linear problem we define the errors

$$\boldsymbol{\eta}_u^n := \boldsymbol{u}_h(t_n) - \boldsymbol{w}_{ht}^n \quad \widetilde{\boldsymbol{\eta}}_u^n := \boldsymbol{u}_h(t_n) - \widetilde{\boldsymbol{w}}_{ht}^n \quad \eta_p^n := p_h(t_n) - r_{ht}^n$$

and the propagation operator $\delta_t a^n := a^n - a^{n-1}$.
For the nonlinear problem we define the errors

$$\boldsymbol{e}_u^n := \boldsymbol{w}_{ht}^n - \boldsymbol{u}_{ht}^n \quad \widetilde{\boldsymbol{e}}_u^n := \widetilde{\boldsymbol{w}}_{ht}^n - \widetilde{\boldsymbol{u}}_{ht}^n \quad e_p^n := r_{ht}^n - p_{ht}^n.$$

Note that it holds $\widetilde{\boldsymbol{\xi}}_u^n = \widetilde{\boldsymbol{\eta}}_u^n + \widetilde{\boldsymbol{e}}_u^n = \boldsymbol{u}_h(t_n) - \widetilde{\boldsymbol{u}}_{ht}^n$, $\boldsymbol{\xi}_u^n = \boldsymbol{\eta}_u^n + \boldsymbol{e}_u^n = \boldsymbol{u}_h(t_n) - \boldsymbol{u}_{ht}^n$ and $\xi_p^n = \eta_p^n + e_p^n = p_h(t_n) - p_{ht}^n$.

3.3.2 Initialization of the Fully Discretized Scheme

We first establish a bound on the initial errors obtained by using the BDF1 scheme (1.2.7).

Lemma 3.3.2. *The initial errors due to time discretization can be bounded by*

$$\|\widetilde{\boldsymbol{\xi}}_u^m\|_0^2 + \nu\Delta t\|\widetilde{\boldsymbol{\xi}}_u^m\|_1^2 + (\Delta t)^2\|\nabla\xi_p^m\|_0^2 \le C(\Delta t)^4 \quad \forall m \in \{1,2\}$$

provided the time step satisfies

$$\left(\frac{h^{4k_u} + \nu^2(\Delta t)^{4l}}{(\Delta t)^2\nu^5} + \frac{1}{\nu}\right) + \frac{\max_M\{\tau_M^n\}}{h^d} + \frac{\max_M\{\tau_M^n\}^2}{\nu h^{2d}} \lesssim \Delta t \quad \forall n \in \{1,2\}.$$

Proof. The convection-diffusion error equation corresponding to (1.2.7) reads:

$$\begin{aligned}
&\left(\frac{\widetilde{\boldsymbol{\xi}}_u^1 - \boldsymbol{\xi}_u^0}{\Delta t}, \boldsymbol{v}_h\right) + \nu(\nabla\widetilde{\boldsymbol{\xi}}_u^1, \nabla\boldsymbol{v}_h) + \gamma(\nabla\cdot\widetilde{\boldsymbol{\xi}}_u^1, \nabla\cdot\boldsymbol{v}_h)\\
&\quad + (\nabla(p_h^1 - p_{ht}^0), \boldsymbol{v}_h) + s_h(\widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{\xi}}_u^1, \widetilde{\boldsymbol{u}}_{ht}^1, \boldsymbol{v}_h)\\
&= c(\widetilde{\boldsymbol{u}}_{ht}^1; \widetilde{\boldsymbol{u}}_{ht}^1, \boldsymbol{v}_h) - c(\boldsymbol{u}_h^1; \boldsymbol{u}_h^1, \boldsymbol{v}_h) + s_h(\widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{u}}_{ht}^1, \boldsymbol{v}_h)\\
&\quad - s_h(\boldsymbol{u}_h^1, \boldsymbol{u}_h^1, \boldsymbol{u}_h^1, \boldsymbol{v}_h) + s_h(\widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{\xi}}_u^1, \widetilde{\boldsymbol{u}}_{ht}^1, \boldsymbol{v}_h) + (\boldsymbol{R}^1, \boldsymbol{v}_h).
\end{aligned}$$

where $\boldsymbol{R}^1$ is defined by

$$(\boldsymbol{R}^1, \boldsymbol{v}_h) := \left(\frac{\boldsymbol{u}_h(t_1) - \boldsymbol{u}_h(t_0)}{\Delta t} - \partial_t u(t_1), \boldsymbol{v}_h\right).$$

Testing this equation with $\widetilde{\boldsymbol{\xi}}_u^1$ yields

$$\begin{aligned}
&\|\widetilde{\boldsymbol{\xi}}_u^1\|_0^2 + \nu\Delta t\|\nabla\widetilde{\boldsymbol{\xi}}_u^1\|_0^2 + \gamma\Delta t\|\nabla\cdot\widetilde{\boldsymbol{\xi}}_u^1\|_0^2 + \Delta t\, s_h(\widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{\xi}}_u^1, \widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{\xi}}_u^1)\\
&\le \Delta t(\|\boldsymbol{\xi}_u^0\|_0 + \|\nabla(p_h^0 - p_h^1)\|_0 + \|\boldsymbol{R}^1\|_0)\|\widetilde{\boldsymbol{\xi}}_u^1\|_0\\
&\quad + \Delta t(c(\widetilde{\boldsymbol{u}}_{ht}^1; \widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{\xi}}_u^1) - c(\boldsymbol{u}_h^1; \boldsymbol{u}_h^1, \widetilde{\boldsymbol{\xi}}_u^1) + s_h(\widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{\xi}}_u^1)\\
&\quad - s_h(\boldsymbol{u}_h^1, \boldsymbol{u}_h^1, \boldsymbol{u}_h^1, \widetilde{\boldsymbol{\xi}}_u^1) + s_h(\widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{\xi}}_u^1, \widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{\xi}}_u^1))
\end{aligned} \tag{3.3.4}$$

We estimate the convective terms as follows

$$\begin{aligned}
&c(\widetilde{\boldsymbol{u}}_{ht}^1; \widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{\xi}}_u^1) - c(\boldsymbol{u}_h^1; \boldsymbol{u}_h^1, \widetilde{\boldsymbol{\xi}}_u^1) = -\underbrace{c(\widetilde{\boldsymbol{u}}_{ht}^1; \widetilde{\boldsymbol{\xi}}_u^1, \widetilde{\boldsymbol{\xi}}_u^1)}_{0} - c(\widetilde{\boldsymbol{\xi}}_u^1; \boldsymbol{u}_h^1, \widetilde{\boldsymbol{\xi}}_u^1)\\
&= -c(\widetilde{\boldsymbol{\xi}}_u^1; \boldsymbol{u}^1, \widetilde{\boldsymbol{\xi}}_u^1) + c(\widetilde{\boldsymbol{\xi}}_u^1; \boldsymbol{u}(t_n) - \boldsymbol{u}_h^1, \widetilde{\boldsymbol{\xi}}_u^1)\\
&\le \|\widetilde{\boldsymbol{\xi}}_u^1\|_0\|\boldsymbol{u}^1\|_2\|\widetilde{\boldsymbol{\xi}}_u^1\|_1 + C\|\widetilde{\boldsymbol{\xi}}_u^1\|_0^{1/2}\|\boldsymbol{u}(t_n) - \boldsymbol{u}_h^1\|_1\|\widetilde{\boldsymbol{\xi}}_u^1\|_1^{3/2}\\
&\le \frac{\nu}{2}\|\nabla\widetilde{\boldsymbol{\xi}}_u^1\|_0^2 + C\left(\frac{\|\boldsymbol{u}(t_n) - \boldsymbol{u}_h^1\|_1^4}{\nu^3} + \frac{1}{\nu}\right)\widetilde{\boldsymbol{\xi}}_u^1\|_0^2
\end{aligned}$$

$$\leq \frac{\nu}{2}\|\nabla\widetilde{\boldsymbol{\xi}}_u^1\|_0^2 + C\left(\frac{h^{4k_u} + \nu^2(\Delta t)^{4l}}{(\Delta t)^2\nu^5} + \frac{1}{\nu}\right)\|\widetilde{\boldsymbol{\xi}}_u^1\|_0^2$$

where we use 3.2.4 for a bound of $\|\boldsymbol{u}_h(t_n)\|_1$. For the nonlinear stabilization we obtain

$$\begin{aligned}
&s_h(\widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{\xi}}_u^1) - s_h(\boldsymbol{u}_h^1, \boldsymbol{u}_h^1, \boldsymbol{u}_h^1, \widetilde{\boldsymbol{\xi}}_u^1) + s_h(\widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{\xi}}_u^1, \widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{\xi}}_u^1)\\
&= s_h(\widetilde{\boldsymbol{u}}_{ht}^1, \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{\xi}}_u^1) - s_h(\boldsymbol{u}_h^1, \boldsymbol{u}_h^1, \boldsymbol{u}_h^1, \widetilde{\boldsymbol{\xi}}_u^1)\\
&= -s_h(\widetilde{\boldsymbol{\xi}}_u^1, \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{u}}_{ht}^1, \widetilde{\boldsymbol{\xi}}_u^1) - s_h(\boldsymbol{u}_h(t_n), \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{\xi}}_u^1, \widetilde{\boldsymbol{\xi}}_u^1)\\
&\leq \frac{C}{h^d}\max_M\{\tau_M^1\}\|\widetilde{\boldsymbol{\xi}}_u^1\|_0^2 + \frac{1}{4}\sum_M \tau_M^1\|\widetilde{\boldsymbol{u}}_M^1\cdot\nabla\widetilde{\boldsymbol{\xi}}_u^1\|_{0,M}^2\\
&\quad + \frac{\nu}{8}\|\widetilde{\boldsymbol{\xi}}_u^1\|_1^2 + \frac{C\max_M\{\tau_M^1\}^2}{\nu h^{2d}}\|\widetilde{\boldsymbol{\xi}}_u^1\|_0^2
\end{aligned}$$

In combination this means

$$\begin{aligned}
&\left(1 - C\Delta t\left(\frac{h^{4k_u} + \nu^2(\Delta t)^{4l}}{(\Delta t)^2\nu^5} + \frac{1}{\nu} - \frac{\max_M\{\tau_M^1\}}{\nu h^d} - \frac{\max_M\{\tau_M^1\}^2}{\nu h^{2d}}\right)\right)\|\widetilde{\boldsymbol{\xi}}_u^1\|_0^2\\
&\quad + \nu\Delta t\|\nabla\widetilde{\boldsymbol{\xi}}_u^1\|_0 + \gamma\Delta t\|\nabla\cdot\widetilde{\boldsymbol{\xi}}_u^1\|_0 + \Delta t\tau_M^1\|(\widetilde{\boldsymbol{u}}_{ht}^1\cdot\nabla)\widetilde{\boldsymbol{\xi}}_u^1\|_0^2\\
&\leq C(\Delta t)^2
\end{aligned}$$

Next we consider the error equation due to the projection step in (1.2.7)

$$\left(\frac{\boldsymbol{\xi}_u^1 - \widetilde{\boldsymbol{\xi}}_u^1}{\Delta t}, \nabla q_h\right) + (\nabla(p_h(t_1) - p_{ht}^1), \nabla q_h) = (\nabla(p_h(t_1) - p_h(t_0)), \nabla q_h).$$

Choosing $q_h = p_h(t_1) - p_{ht}^1$ we arrive at

$$\begin{aligned}
&\Delta t\|\nabla(p_h(t_1) - p_{ht}^1)\|_0^2\\
&\leq \|\widetilde{\boldsymbol{\xi}}_u^1\|_0 + \Delta t\|(\nabla(p_h(t_0) - p_h(t_1))\|_0)\|\nabla(p_h(t_1) - p_{ht}^1)\|_0
\end{aligned} \tag{3.3.5}$$

where we used that $\boldsymbol{\xi}_u$ is weakly solenoidal. Hence $\|\nabla(p_h(t_1) - p_{ht}^1)\|_0 \leq C\Delta t$ holds. Testing (3.3.5) with $\boldsymbol{\xi}_u^1$ gives

$$\|\boldsymbol{\xi}_u^1\|_0^2 \leq \|\widetilde{\boldsymbol{\xi}}_u^1\|_0 + \Delta t\|(\nabla(p_h(t_0) - p_h(t_1))\|_0)\|\boldsymbol{\xi}_u^1\|_0 \tag{3.3.6}$$

and finally $\|\boldsymbol{\xi}_u^1\|_0 \leq C(\Delta t)^2$.

Next, we need an estimate for $\widetilde{\boldsymbol{\xi}}_u^2$. Applying the same technique for $n = 2$

with the abbreviation $\boldsymbol{R}^2 := D_t \boldsymbol{u}_h(t_2) - \partial_t \boldsymbol{u}_h(t_2)$ gives

$$
\begin{aligned}
&\left(\frac{3\widetilde{\boldsymbol{\xi}}_u^2 - 3\widetilde{\boldsymbol{\xi}}_u^1}{2\Delta t}, \widetilde{\boldsymbol{\xi}}_u^2\right) + \nu(\nabla(\widetilde{\boldsymbol{\xi}}_u^2 - \widetilde{\boldsymbol{\xi}}_u^1), \nabla\widetilde{\boldsymbol{\xi}}_u^2) \\
&= (\boldsymbol{R}^2, \widetilde{\boldsymbol{\xi}}_u^2) + \nabla(p_{ht}^1 - p_h(t_2), \widetilde{\boldsymbol{\xi}}_u^2) - \nu(\nabla(\widetilde{\boldsymbol{\xi}}_u^1 - \widetilde{\boldsymbol{\xi}}_u^0), \nabla\widetilde{\boldsymbol{\xi}}_u^2) \\
&\quad + \left(\frac{3\boldsymbol{\xi}_u^1 - 3\widetilde{\boldsymbol{\xi}}_u^1}{2\Delta t}, \widetilde{\boldsymbol{\xi}}_u^2\right) + \left(\frac{\boldsymbol{\xi}_u^1 - \boldsymbol{\xi}_u^0}{2\Delta t}, \widetilde{\boldsymbol{\xi}}_u^2\right) \\
&= (\boldsymbol{R}^2, \widetilde{\boldsymbol{\xi}}_u^2) + \nabla(p_{ht}^1 - p(t_2), \widetilde{\boldsymbol{\xi}}_u^2) - \nu(\nabla(\widetilde{\boldsymbol{\xi}}_u^1 - \widetilde{\boldsymbol{\xi}}_u^0), \nabla\widetilde{\boldsymbol{\xi}}_u^2) \\
&\quad + \frac{3}{2}(\nabla(p_{ht}^1 - p_{ht}^0), \widetilde{\boldsymbol{\xi}}_u^2) + \left(\frac{\boldsymbol{\xi}_u^1 - \boldsymbol{\xi}_u^0}{2\Delta t}, \widetilde{\boldsymbol{\xi}}_u^2\right) \\
&\quad + \Delta t(c(\widetilde{\boldsymbol{u}}_{ht}^2; \widetilde{\boldsymbol{u}}_{ht}^2, \Delta\widetilde{\boldsymbol{\xi}}_u^2) - c(\boldsymbol{u}_h^2; \boldsymbol{u}_h^2, \widetilde{\boldsymbol{\xi}}_u^2)) \\
&= (\boldsymbol{R}^2, \widetilde{\boldsymbol{\xi}}_u^2) - \nu(\nabla(\widetilde{\boldsymbol{\xi}}_u^1 - \widetilde{\boldsymbol{\xi}}_u^0), \nabla\widetilde{\boldsymbol{\xi}}_u^2) + \left(\frac{\boldsymbol{\xi}_u^1 - \boldsymbol{\xi}_u^0}{2\Delta t}, \widetilde{\boldsymbol{\xi}}_u^2\right) \\
&\quad + \left(\nabla\left(\frac{5}{2}p_{ht}^1 - \frac{3}{2}p_{ht}^0 - p_h(t_2)\right), \widetilde{\boldsymbol{\xi}}_u^2\right) \\
&\quad + \Delta t(c(\widetilde{\boldsymbol{u}}_{ht}^2; \widetilde{\boldsymbol{u}}_{ht}^2, \Delta\widetilde{\boldsymbol{\xi}}_u^2) - c(\boldsymbol{u}_h^2; \boldsymbol{u}_h^2, \widetilde{\boldsymbol{\xi}}_u^2)) \\
&= (\boldsymbol{R}^2, \widetilde{\boldsymbol{\xi}}_u^2) - \nu(\nabla(\widetilde{\boldsymbol{\xi}}_u^1 - \widetilde{\boldsymbol{\xi}}_u^0), \nabla\widetilde{\boldsymbol{\xi}}_u^2) + \left(\frac{\boldsymbol{\xi}_u^1 - \boldsymbol{\xi}_u^0}{2\Delta t}, \widetilde{\boldsymbol{\xi}}_u^2\right) \\
&\quad + \frac{5}{2}(\nabla(p_{ht}^1 - p_h(t_1)), \widetilde{\boldsymbol{\xi}}_u^2) \\
&\quad + \left(\nabla\left(\frac{5}{2}p_h(t_1) - \frac{3}{2}p_h(t_0) - p_h(t_2)\right), \widetilde{\boldsymbol{\xi}}_u^2\right) \\
&\quad + \Delta t(c(\widetilde{\boldsymbol{u}}_{ht}^2; \widetilde{\boldsymbol{u}}_{ht}^2, \widetilde{\boldsymbol{\xi}}_u^2) - c(\boldsymbol{u}_h^2; \boldsymbol{u}_h^2, \widetilde{\boldsymbol{\xi}}_u^2)) \\
&\leq \min\{C\Delta t\|\widetilde{\boldsymbol{\xi}}_u^2\|_0, C\Delta t\|\widetilde{\boldsymbol{\xi}}_u^2\|_1\} \\
&\Rightarrow \|\widetilde{\boldsymbol{\xi}}_u^2 - \widetilde{\boldsymbol{\xi}}_u^1\|_0^2 \leq C(\Delta t)^4 \qquad \nu\|\widetilde{\boldsymbol{\xi}}_u^2 - \widetilde{\boldsymbol{\xi}}_u^1\|_1^2 \leq C(\Delta t)^3
\end{aligned}
$$

Since the error $\boldsymbol{\xi}_u^1$ is an orthogonal $L^2(\Omega)$ projection of $\widetilde{\boldsymbol{\xi}}_u^1$ we also get

$$
\|\boldsymbol{\xi}_u^2 - \boldsymbol{\xi}_u^1\|_0 \leq \|\widetilde{\boldsymbol{\xi}}_u^2 - \widetilde{\boldsymbol{\xi}}_u^1\|_0 \leq C(\Delta t)^2
$$

For the pressure error we again use the projection equation

$$
\begin{aligned}
(\nabla(\xi_p^1 - \xi_p^2), \nabla q_h) &= (\nabla(p_h(t_1) - p_h(t_2)), \nabla q_h) - \left(\frac{3\boldsymbol{\xi}_u^2 - 3\widetilde{\boldsymbol{\xi}}_u^2}{2\Delta t}, \nabla q_h\right) \\
&\leq C\Delta t\|\nabla q_h\|_0 \\
\Rightarrow \|\nabla\xi_p^2\|_0 &\leq \|\nabla\xi_p^1\|_0 + \|\nabla(\xi_p^1 - \xi_p^2)\|_0 \leq C\Delta t.
\end{aligned}
\tag{3.3.7}
$$

□

3.3.3 Error Estimates for the Linear Auxiliary Problem

The auxiliary problem is handled in two steps. First initial errors are considered and afterwards estimates for $n \geq 3$ are derived.

Initialization of the Auxiliary Problem

For initializing the algorithm we use a BDF1-scheme defined as follows

Find $\widetilde{\boldsymbol{w}}_{ht}^1 \in \boldsymbol{V}_h, \boldsymbol{w}_{ht}^1 \in \boldsymbol{Y}_h$ and $r_{ht}^1 \in Q_h$ such that

$$\begin{aligned}
&\left(\frac{\widetilde{\boldsymbol{w}}_{ht}^1 - \boldsymbol{u}_h(t_0)}{\Delta t}, \boldsymbol{v}_h\right) + \nu(\nabla \widetilde{\boldsymbol{w}}_{ht}^1, \nabla \boldsymbol{v}_h) + \gamma(\nabla \cdot \widetilde{\boldsymbol{w}}_{ht}^1, \nabla \cdot \boldsymbol{v}_h) \\
&= (-\nabla p_h(t_0) + \boldsymbol{f}^1, \boldsymbol{v}_h) - c(\boldsymbol{u}_h^1; \boldsymbol{u}_h^1, \boldsymbol{v}_h) - s_h(\boldsymbol{u}_h^1; \boldsymbol{u}_h^1, \boldsymbol{v}_h) \\
&\widetilde{\boldsymbol{w}}_{ht}^1|_{\partial\Omega} = 0
\end{aligned} \tag{3.3.8}$$

$$\begin{aligned}
\left(\frac{\boldsymbol{w}^1 - \widetilde{\boldsymbol{w}}^1}{\Delta t}, \nabla q_h\right) + (\nabla(r_{ht}^1 - p_h(t_0)), \nabla y_h) = 0. \\
(\nabla \cdot \boldsymbol{w}_h^1, q_h) = 0 \\
(\boldsymbol{n} \cdot \boldsymbol{w}_{ht}^1)|_{\partial\Omega} = 0
\end{aligned} \tag{3.3.9}$$

holds for all $\boldsymbol{v}_h \in \boldsymbol{V}_h, \boldsymbol{y}_h \in \boldsymbol{Y}_h$ and $q_h \in Q_h$.

For the linear auxiliary problem we now perform estimates similar to the those for the initial errors of the full discretization in time.

Lemma 3.3.3. *The initial errors for the linear auxiliary problem can be bounded by*

$$\|\widetilde{\boldsymbol{\eta}}_u^m\|_0^2 + \nu\Delta t\|\widetilde{\boldsymbol{\eta}}_u^m\|_1^2 + (\Delta t)^2\|\nabla \eta_p^m\|_0^2 \leq C(\Delta t)^4 \quad \forall m \in \{1, 2\}.$$

Proof. The error equation corresponding to the convection-diffusion step reads:

$$\begin{aligned}
&\left(\frac{\widetilde{\boldsymbol{\eta}}_u^1}{\Delta t}, \boldsymbol{v}_h\right) + \nu(\nabla \widetilde{\boldsymbol{\eta}}_u^1, \nabla \boldsymbol{v}_h) + \gamma(\nabla \cdot \widetilde{\boldsymbol{\eta}}_u^1, \nabla \cdot \boldsymbol{v}_h) \\
&\quad + (\nabla(p_h(\Delta t) - p_h(t_0)), \boldsymbol{v}_h) \\
&= \left(\frac{\boldsymbol{u}_h(t_1) - \boldsymbol{u}_h(t_0)}{\Delta t} - \partial_t \boldsymbol{u}_h(t_1), \boldsymbol{v}_h\right) =: (\boldsymbol{R}^1, \boldsymbol{v}_h)
\end{aligned} \tag{3.3.10}$$

Testing this equation with $\widetilde{\boldsymbol{\eta}}_u^1$ yields

$$\begin{aligned}
&\|\widetilde{\boldsymbol{\eta}}_u^1\|_0^2 + \nu\Delta t\|\nabla \widetilde{\boldsymbol{\eta}}_u^1\|_0^2 + \gamma\Delta t\|\nabla \cdot \widetilde{\boldsymbol{\eta}}_u^1\|_0^2 \\
&\leq \Delta t(\|\nabla(p_h(t_0) - p_h(\Delta t))\|_0 + \|\boldsymbol{R}^1\|_0)\|\widetilde{\boldsymbol{\eta}}_u^1\|_0 \\
&\leq C(\Delta t)^2\|\widetilde{\boldsymbol{\eta}}_u^1\|_0
\end{aligned} \tag{3.3.11}$$

and hence $\|\widetilde{\boldsymbol{\eta}}_u^1 - \widetilde{\boldsymbol{\eta}}_u^0\|_0^2 = \|\widetilde{\boldsymbol{\eta}}_u^1\|_0^2 \le C(\Delta t)^4$ and $\nu\|\nabla\widetilde{\boldsymbol{\eta}}_u^1\|_0^2 \le C(\Delta t)^3$.

Next we consider the error equation due to the projection step (3.3.9)

$$\left(\frac{\boldsymbol{\eta}_u^1 - \widetilde{\boldsymbol{\eta}}_u^1}{\Delta t}, \nabla q_h\right) + (\nabla(p_h(\Delta t) - r_{ht}^1), \nabla q_h) = (\nabla(p_h(t_1) - p_h(t_0)), \nabla q_h).$$

Choosing $q_h = p_h(t_1) - r_{ht}^1$ we arrive at

$$\begin{aligned}&\Delta t\|\nabla(p_h(t_1) - r_{ht}^1)\|_0^2\\ &\le (\|\widetilde{\boldsymbol{\eta}}_u^1\|_0 + \Delta t\|\nabla(p_h(t_0) - p_h(\Delta t))\|_0)\|\nabla(p_h(t_1) - r_{ht}^1)\|_0\end{aligned}$$

where we used that $\boldsymbol{\eta}_u$ is weakly solenoidal. Hence $\|\nabla(p_h(t_1) - r_{ht}^1)\|_0 \le C\Delta t$ holds.

Next, we need an estimate for $\widetilde{\boldsymbol{\eta}}_u^2$. Applying the same technique for $n = 2$ with the abbreviation $\boldsymbol{R}^2 := D_t\boldsymbol{u}_h(t_2) - \partial_t\boldsymbol{u}_h(t_2)$ gives

$$\begin{aligned}
&\left(\frac{3\widetilde{\boldsymbol{\eta}}_u^2 - 3\widetilde{\boldsymbol{\eta}}_u^1}{2\Delta t}, \boldsymbol{v}_h\right) + \nu(\nabla(\widetilde{\boldsymbol{\eta}}_u^2 - \widetilde{\boldsymbol{\eta}}_u^1), \nabla\boldsymbol{v}_h)\\
&= (\boldsymbol{R}^2, \boldsymbol{v}_h) + \nabla(r^1 - p_h(t_2), \boldsymbol{v}_h) - \nu(\nabla(\widetilde{\boldsymbol{\eta}}_u^1 - \widetilde{\boldsymbol{\eta}}_u^0), \nabla\boldsymbol{v}_h)\\
&\quad + \left(\frac{3\boldsymbol{\eta}_u^1 - 3\widetilde{\boldsymbol{\eta}}_u^1}{2\Delta t}, \boldsymbol{v}_h\right) + \left(\frac{\boldsymbol{\eta}_u^1 - \boldsymbol{\eta}_u^0}{2\Delta t}, \boldsymbol{v}_h\right)\\
&= (\boldsymbol{R}^2, \boldsymbol{v}_h) + \nabla(r^1 - p(t_2), \boldsymbol{v}_h) - \nu(\nabla(\widetilde{\boldsymbol{\eta}}_u^1 - \widetilde{\boldsymbol{\eta}}_u^0), \nabla\boldsymbol{v}_h)\\
&\quad + \frac{3}{2}(\nabla(r^1 - r^0), \boldsymbol{v}_h) + \left(\frac{\boldsymbol{\eta}_u^1 - \boldsymbol{\eta}_u^0}{2\Delta t}, \boldsymbol{v}_h\right)\\
&= (\boldsymbol{R}^2, \boldsymbol{v}_h) - \nu(\nabla(\widetilde{\boldsymbol{\eta}}_u^1 - \widetilde{\boldsymbol{\eta}}_u^0), \nabla\boldsymbol{v}_h) + \left(\frac{\boldsymbol{\eta}_u^1 - \boldsymbol{\eta}_u^0}{2\Delta t}, \boldsymbol{v}_h\right)\\
&\quad + \left(\nabla\left(\frac{5}{2}r^1 - \frac{3}{2}r^0 - p_h(t_2)\right), \boldsymbol{v}_h\right)\\
&= (\boldsymbol{R}^2, \boldsymbol{v}_h) - \nu(\nabla(\widetilde{\boldsymbol{\eta}}_u^1 - \widetilde{\boldsymbol{\eta}}_u^0), \nabla\boldsymbol{v}_h) + \left(\frac{\boldsymbol{\eta}_u^1 - \boldsymbol{\eta}_u^0}{2\Delta t}, \boldsymbol{v}_h\right)\\
&\quad + \frac{5}{2}(\nabla(r^1 - p_h(t_1)), \boldsymbol{v}_h)\\
&\quad + \left(\nabla\left(\frac{5}{2}p_h(t_1) - \frac{3}{2}p_h(t_0) - p_h(t_2)\right), \boldsymbol{v}_h\right)\\
&\le \min\{C\Delta t\|\boldsymbol{v}_h\|_0, C\Delta t\|\boldsymbol{v}_h\|_1\}\\
&\Rightarrow \|\widetilde{\boldsymbol{\eta}}_u^2 - \widetilde{\boldsymbol{\eta}}_u^1\|_0^2 \le C(\Delta t)^4 \qquad \nu\|\widetilde{\boldsymbol{\eta}}_u^2 - \widetilde{\boldsymbol{\eta}}_u^1\|_1^2 \le C(\Delta t)^3.
\end{aligned} \tag{3.3.12}$$

Since the error ξ^1 is an orthogonal $L^2(\Omega)$ projection of $\widetilde{\xi}^1$ we also get

$$\|\boldsymbol{\eta}_u^2 - \boldsymbol{\eta}_u^1\|_0 \le \|\widetilde{\boldsymbol{\eta}}_u^2 - \widetilde{\boldsymbol{\eta}}_u^1\|_0 \le C(\Delta t)^2$$

For the pressure error we again use the projection equation

$$
\begin{aligned}
&(\nabla((p_h(t_1) - r_{ht}^1) - (p_h(t_2) - r_{ht}^2)), \nabla q_h) \\
&= (\nabla(p_h(t_1) - p_h(t_2)), \nabla q_h) - \left(\frac{3\boldsymbol{\eta}_u^2 - 3\widetilde{\boldsymbol{\eta}}_u^2}{2\Delta t}, \nabla q_h \right) \leq C\Delta t \|\nabla q_h\|_0 \\
\Rightarrow &\|\nabla(p_h(t_2) - r_{ht}^2)\|_0 \\
&\leq \|\nabla(p_h(t_1) - r_{ht}^1)\|_0 + \|\nabla((p_h(t_1) - r_{ht}^1) - (p_h(t_2) - r_{ht}^2))\|_0 \\
&\leq C\Delta t.
\end{aligned}
\tag{3.3.13}
$$

□

Error Estimates for the Velocity

Now, we are in position to derive estimates for the auxiliary problems when $n \geq 2$. We aim for the following result:

Theorem 3.3.4. *For all $1 \leq m \leq N$ the velocity error due to discretization in time can be bounded as*

$$
\begin{aligned}
\|\boldsymbol{\eta}_u^m\|_0 + \|\widetilde{\boldsymbol{\eta}}_u^n\|_0 &\leq C\frac{(\Delta t)^2}{\nu} \\
\|\nabla\widetilde{\boldsymbol{\eta}}_u^n\|_0 + \|\boldsymbol{\eta}_p^n\|_0 &\leq C\Delta t
\end{aligned}
\tag{3.3.14}
$$

A first result is concerned with the difference between the solenoidal and the non-solenoidal error.

Lemma 3.3.5. *For all $1 \leq m \leq N$ the difference between the velocity errors can be bounded as*

$$\|\boldsymbol{\eta}_u^m - \widetilde{\boldsymbol{\eta}}_u^m\|_0^2 \leq C_{G,lin}(\Delta t)^4.$$

with a Gronwall term $C_{G,lin} \sim \exp\big(T(1-2\Delta t)^{-1}\big)$.

Proof. The error equation due to the convection-diffusion step (3.3.2) reads

$$
\begin{aligned}
&\left(\frac{3\widetilde{\boldsymbol{\eta}}_u^n - 4\boldsymbol{\eta}_u^{n-1} + \boldsymbol{\eta}_u^{n-2}}{2\Delta t}, \boldsymbol{v}_h \right) + \nu(\nabla\widetilde{\boldsymbol{\eta}}_u^n, \nabla\boldsymbol{v}_h) + \gamma(\nabla\cdot\widetilde{\boldsymbol{\xi}}^n, \nabla\cdot\boldsymbol{v}_h) \\
&= (\boldsymbol{R}^n, \boldsymbol{v}_h) - (\nabla(p_h(t_n) - r_{ht}^{n-1}), \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h
\end{aligned}
\tag{3.3.15}
$$

where the residuum $\boldsymbol{R}^n$ is given by

$$(\boldsymbol{R}^n, \boldsymbol{v}_h) = (D_t\boldsymbol{u}_h(t_n) - \partial_t\boldsymbol{u}_h(t_n), \boldsymbol{v}_h).$$

Since the propagation operator is linear we also get

$$
\begin{aligned}
&\left(\frac{3\delta_t\widetilde{\boldsymbol{\eta}}_u^n - 4\delta_t\boldsymbol{\eta}_u^{n-1} + \delta_t\boldsymbol{\eta}_u^{n-2}}{2\Delta t}, \boldsymbol{v}_h \right) + \nu(\nabla\delta_t\widetilde{\boldsymbol{\eta}}_u^n, \nabla\boldsymbol{v}_h) \\
&\quad + \gamma(\nabla\cdot\delta_t\widetilde{\boldsymbol{\eta}}_u^n, \nabla\cdot\boldsymbol{v}_h) \\
&= (\delta_t\boldsymbol{R}^n, \boldsymbol{v}_h) - (\nabla\delta_t(p_h(t_n) - r_{ht}^{n-1}), \boldsymbol{v}_h).
\end{aligned}
\tag{3.3.16}
$$

Now, we can do the same for the error corresponding to the projection step (3.3.3) and get

$$\left(\frac{3\boldsymbol{\eta}_u^n - 3\widetilde{\boldsymbol{\eta}}_u^n}{2\Delta t} + \nabla \eta_p^n - \nabla(p_h(t_n) - r_{ht}^{n-1}), \nabla q_h\right) = 0 \tag{3.3.17}$$

$$\left(\frac{3\delta_t\boldsymbol{\eta}_u^n - 3\delta_t\widetilde{\boldsymbol{\eta}}_u^n}{2\Delta t} + \nabla \delta_t\eta_p^n - \nabla\delta_t(p_h(t_n) - r_{ht}^{n-1}), \nabla q_h\right) = 0. \tag{3.3.18}$$

Testing the incremental error equation (3.3.16) with $4\Delta t \delta_t \widetilde{\boldsymbol{\eta}}_u^n$ we arrive at

$$\begin{aligned}
&(2(3\delta_t\widetilde{\boldsymbol{\eta}}_u^n - 4\delta_t\boldsymbol{\eta}_u^{n-1} + \delta_t\boldsymbol{\eta}_u^{n-2}), \delta_t\widetilde{\boldsymbol{\eta}}_u^n) \\
&\quad + 4\Delta t\nu\|\nabla\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + 4\Delta t\gamma\|\nabla\cdot\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 \\
&\quad = 4\Delta t(\delta_t\boldsymbol{R}^n - \nabla\delta_t(p_h(t_n) - r_{ht}^{n-1}), \delta_t\widetilde{\boldsymbol{\eta}}_u^n).
\end{aligned} \tag{3.3.19}$$

The first term is then split according to (cf. A.1.1)

$$\begin{aligned}
&(2(3\delta_t\widetilde{\boldsymbol{\eta}}_u^n - 4\delta_t\boldsymbol{\eta}_u^{n-1} + \delta_t\boldsymbol{\eta}_u^{n-2}), \delta_t\widetilde{\boldsymbol{\eta}}_u^n) = I_1 + I_2 + I_3 \\
&I_1 := 3\|\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + 3\|\delta_t\boldsymbol{\eta}_u^n - \delta_t\widetilde{\boldsymbol{\eta}_u}^n\|_0^2 - 3\|\delta_t\boldsymbol{\eta}_u^n\|_0^2 \\
&I_2 := 2(\delta_t\widetilde{\boldsymbol{\eta}}_u^n - \delta_t\boldsymbol{\eta}_u^n, 3\delta_t\boldsymbol{\eta}_u^n - 4\delta_t\boldsymbol{\eta}_u^{n-1} + \delta_t\boldsymbol{\eta}_u^{n-2}) \\
&I_3 := \|\delta_t\boldsymbol{\eta}_u^n\|_0^2 + \|2\delta_t\boldsymbol{\eta}_u^n - \delta_t\boldsymbol{\eta}_u^{n-1}\|_0^2 + \|\delta_{ttt}\boldsymbol{\eta}_u^n\|_0^2 \\
&\qquad - \|\delta_t\boldsymbol{\eta}_u^{n-1}\|_0^2 - \|2\delta_t\boldsymbol{\eta}_u^{n-1} - \delta_t\boldsymbol{\eta}_u^{n-2}\|_0^2
\end{aligned}$$

and the second term vanishes

$$\begin{aligned}
\frac{3}{4\Delta t}I_2 &= (\nabla((\delta_t(p_h(t_n) - r_{ht}^{n-1}) - \delta_t\eta_p^n), 3\delta_t\boldsymbol{\eta}_u^n - 4\delta_t\boldsymbol{\eta}_u^{n-1} + \delta_t\boldsymbol{\eta}_u^{n-2}) \\
&= -(\delta_t(p_h(t_n) - r_{ht}^{n-1}) - \delta_t\eta_p^n), \nabla\cdot(3\delta_t\boldsymbol{\eta}_u^n - 4\delta_t\boldsymbol{\eta}_u^{n-1} + \delta_t\boldsymbol{\eta}_u^{n-2})) \\
&= 0
\end{aligned}$$

due to the fact that $\boldsymbol{u}_h$ and $\boldsymbol{w}^n$ are weakly divergence-free.

Now, we test the error in the projection step (3.3.18) with $\delta_t\eta_p^{n-1} = \delta_t(p_h(t_{n-1}) - r_{ht}^{n-1})$ and get after integration by parts for the first term

$$\begin{aligned}
&-\left(\frac{3}{2\Delta t}\delta_t\widetilde{\boldsymbol{\eta}}_u^n, \nabla\delta_t(p_h(t_{n-1}) - r_t^{n-1})\right) \\
&= -(\nabla\delta_t(\eta_p^n - \eta_p^{n-1}), \nabla\delta_t\eta_p^{n-1}) + (\nabla\delta_t(p_h(t_n) - p_h(t_{n-1})), \nabla\delta_t\eta_p^{n-1}) \\
&= -\frac{1}{2}(\|\nabla\delta_t\eta_p^n\|_0^2 - \|\nabla\delta_t(\eta_p^n - \eta_p^{n-1})\|_0^2 - \|\nabla\delta_t\eta_p^{n-1}\|_0^2) \\
&\quad + (\nabla\delta_t(p_h(t_n) - p_h(t_{n-1})), \nabla\delta_t\eta_p^{n-1})
\end{aligned}$$

Using the projection step (3.3.18) again and testing with $\delta_t(\eta_p^n - \eta_p^{n-1})$ we derive

$$
\begin{aligned}
&\frac{2\Delta t}{3}\|\nabla\delta_t(\eta_p^n - \eta_p^{n-1})\|_0^2 \\
&\leq \|\delta_t\boldsymbol{\eta}_u^n - \delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0\|\nabla\delta_t(\eta_p^n - \eta_p^{n-1})\|_0 \\
&\quad + \frac{2\Delta t}{3}(\nabla\delta_t(p_h(t_n) - p_h(t_{n-1})), \nabla\delta_t(\eta_p^n - \eta_p^{n-1})) \\
&\leq \frac{3}{4\Delta t}\|\delta_t\boldsymbol{\eta}_u^n - \delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + \frac{\Delta t}{3}\|\nabla\delta_t(\eta_p^n - \eta_p^{n-1})\|_0^2 \\
&\quad + \frac{2\Delta t}{3}(\nabla\delta_{tt}p_h(t_n), \nabla\delta_t(\eta_p^n - \eta_p^{n-1}))
\end{aligned}
\tag{3.3.20}
$$

and thus

$$
\begin{aligned}
&- 4\Delta t\left(\delta_t\widetilde{\boldsymbol{\eta}}_u^n, \nabla\delta_t(p_h(t_{n-1}) - r_t^{n-1})\right) \\
&\leq \frac{8}{3}(\Delta t)^2(\nabla\delta_t(p_h(t_n) - p_h(t_{n-1})), \nabla\delta_t\eta_p^{n-1}) \\
&\quad - \frac{4}{3}(\Delta t)^2\|\nabla\delta_t\eta_p^n\|_0^2 + \frac{4}{3}(\Delta t)^2\|\nabla\delta_t\eta_p^{n-1}\|_0^2 \\
&\quad + 3\|\delta_t\boldsymbol{\eta}_u^n - \delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + \frac{8}{3}(\Delta t)^2(\nabla\delta_{tt}p_h(t_n), \nabla\delta_t(\eta_p^n - \eta_p^{n-1})).
\end{aligned}
$$

Combining all the estimates gives

$$
\begin{aligned}
&3\|\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + 3\|\delta_t\boldsymbol{\eta}_u^n - \delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 - 3\|\delta_t\boldsymbol{\eta}_u^n\|_0^2 \\
&\quad + \|\delta_t\boldsymbol{\eta}_u^n\|_0^2 + \|2\delta_t\boldsymbol{\eta}_u^n - \delta_t\boldsymbol{\eta}_u^{n-1}\|_0^2 + \|\delta_{ttt}\boldsymbol{\eta}_u^n\|_0^2 \\
&\quad - \|\delta_t\boldsymbol{\eta}_u^{n-1}\|_0^2 - \|2\delta_t\boldsymbol{\eta}_u^{n-1} - \delta_t\boldsymbol{\eta}_u^{n-2}\|_0^2 \\
&\quad + 4\Delta t\nu\|\nabla\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + 4\Delta t\gamma\|\nabla\cdot\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + \frac{4(\Delta t)^2}{3}\|\nabla\delta_t\eta_p^n\|_0^2 \\
&\leq 4\Delta t(\delta_t\boldsymbol{R}^n, \delta_t\widetilde{\boldsymbol{\eta}}_u^n) + \frac{8}{3}(\Delta t)^2(\nabla\delta_t(p_h(t_n) - p_h(t_{n-1})), \nabla\delta_t\eta_p^{n-1}) \\
&\quad + \frac{4}{3}(\Delta t)^2\|\nabla\delta_t\eta_p^{n-1}\|_0^2 \\
&\quad + 3\|\delta_t\boldsymbol{\eta}_u^n - \delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + \frac{8}{3}(\Delta t)^2(\nabla\delta_{tt}p_h(t_n), \nabla\delta_t(\eta_p^n - \eta_p^{n-1}))
\end{aligned}
$$

and using $\|\delta_t\boldsymbol{\eta}_u^n\|_0 \overset{1.2.1}{\leq} \|\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0$ we derive

$$
\begin{aligned}
&\|\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + \|2\delta_t\boldsymbol{\eta}_u^n - \delta_t\boldsymbol{\eta}_u^{n-1}\|_0^2 \\
&\quad + \|\delta_{ttt}\boldsymbol{\eta}_u^n\|_0^2 - \|\delta_t\widetilde{\boldsymbol{\eta}}_u^{n-1}\|_0^2 - \|2\delta_t\boldsymbol{\eta}_u^{n-1} - \delta_t\boldsymbol{\eta}_u^{n-2}\|_0^2 \\
&\quad + 4\Delta t\nu\|\nabla\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + 4\Delta t\gamma\|\nabla\cdot\delta_t\widetilde{\boldsymbol{\eta}}^n\|_0^2 + \frac{4}{3}(\Delta t)^2\|\nabla\delta_t\eta_p^n\|_0^2 \\
&\leq \frac{4}{3}(\Delta t)^2\|\nabla\delta_t\eta_p^{n-1}\|_0^2 + 4\Delta t(\delta_t\boldsymbol{R}^n, \delta_t\widetilde{\boldsymbol{\eta}}_u^n) + 4\Delta t(\nabla\delta_{tt}p_h(t_n), \delta_t\widetilde{\boldsymbol{\eta}}_u^n) \\
&\quad + \frac{8}{3}(\Delta t)^2(\nabla\delta_{tt}p_h(t_n), \nabla\delta_t\eta_p^n)
\end{aligned}
$$

$$\leq \frac{4}{3}(\Delta t)^2\|\nabla\delta_t\eta_p^{n-1}\|_0^2 + 4\Delta t\|\delta_t\boldsymbol{R}^n\|_0^2 + \Delta t\|\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2$$
$$+ 4\Delta t\|\nabla\delta_{tt}p_h(t_n)\|_0^2 + \Delta t\|\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + \frac{8}{3}\Delta t\|\nabla\delta_{tt}p_h(t_n)\|_0^2$$
$$+ \frac{2}{3}(\Delta t)^3\|\nabla\delta_t\eta_p^n\|_0^2.$$

Next, we sum this equation up from $n = 3$ to $m \leq N$:

$$\begin{aligned}
&\|\delta_t\widetilde{\boldsymbol{\eta}}_u^m\|_0^2 + \|2\delta_t\boldsymbol{\eta}_u^m - \delta_t\boldsymbol{\eta}_u^{m-1}\|_0^2 + \frac{4}{3}(\Delta t)^2\|\nabla\delta_t\eta_p^m\|_0^2 \\
&\quad + \sum_{n=3}^{m}\left(\|\delta_{ttt}\boldsymbol{\eta}_u^n\|_0^2 + 4\Delta t\nu\|\nabla\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + 4\Delta t\gamma\|\nabla\cdot\delta_t\widetilde{\boldsymbol{\eta}}^n\|_0^2\right) \\
&\leq \|\delta_t\widetilde{\boldsymbol{\eta}}_u^2\|_0^2 + \|2\delta_t\boldsymbol{\eta}_u^2 - \delta_t\boldsymbol{\eta}_u^1\|_0^2 + \frac{4}{3}(\Delta t)^2\|\nabla\delta_t\eta_p^2\|_0^2 \\
&\quad + \Delta t\sum_{n=3}^{m}\left(2\|\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + \frac{2}{3}(\Delta t)^2\|\nabla\delta_t\eta_p^n\|_0^2\right) \\
&\quad + \Delta t\sum_{n=3}^{m}\left(4\|\delta_t\boldsymbol{R}^n\|_0^2 + 7\|\partial_t^2 p_h(t_n)\|_1^2\right).
\end{aligned}$$

If $2\Delta t \leq 1$ the discrete Gronwall lemma (A.5.1) can be applied to $\|\delta_t\widetilde{\boldsymbol{\eta}}_u^m\|_0^2 + \frac{4}{3}(\Delta t)^2\|\nabla\delta_t\eta_p^m\|_0^2$. In combination with the initial error estimates (3.3.3) and $\|\delta_t\boldsymbol{R}^n\|_0^2 \leq C(\Delta t)^4$ this yields

$$\begin{aligned}
&\|\delta_t\widetilde{\boldsymbol{\eta}}_u^m\|_0^2 + \|2\delta_t\boldsymbol{\eta}_u^m - \delta_t\boldsymbol{\eta}_u^{m-1}\|_0^2 + \frac{4}{3}(\Delta t)^2\|\nabla\delta_t\eta_p^m\|_0^2 \\
&\quad + \sum_{n=3}^{m}\left(\|\delta_{ttt}\boldsymbol{\eta}_u^n\|_0^2 + 4\Delta t\nu\|\nabla\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + 4\Delta t\gamma\|\nabla\cdot\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2\right) \\
&\leq \|\delta_t\widetilde{\boldsymbol{\eta}}_u^2\|_0^2 + \|2\delta_t\boldsymbol{\eta}_u^2 - \delta_t\boldsymbol{\eta}_u^1\|_0^2 + \frac{4}{3}(\Delta t)^2\|\nabla\delta_t\eta_p^2\|_0^2 \\
&\quad + C\Delta t\sum_{n=3}^{m}\left(\|\delta_t\boldsymbol{R}^n\|_0^2 + 7\|\partial_t^2 p_h(t_n)\|_1^2(\Delta t)^4\right) \leq C_{G,lin}(\Delta t)^4
\end{aligned} \tag{3.3.21}$$

where $C_{G,lin} \sim \exp\left(T(1-2\Delta t)^{-1}\right)$. The approximations for $\Delta t\sum_{n=1}^{m}\|\delta_t\boldsymbol{R}^n\|_0^2$ and $\|\partial_t^2 p_h(t_n)\|_1$ rely on the Bramble-Hilbert theorem 3.2.1, (generalized) Taylor expansion and Assumption 3.0.5

Finally, the projection error equation (3.3.18) states

$$\begin{aligned}
\|\boldsymbol{\eta}_u^m - \widetilde{\boldsymbol{\eta}}_u^m\|_0 &= \frac{2\Delta t}{3}\|\nabla(\delta_t\eta_p^n - \delta_t p(t_{k+1}))\|_0 \\
&\leq \frac{2\Delta t}{3}\left(\|\nabla\delta_t\eta_p^n\|_0 + \|\delta_t p(t_{k+1})\|_0\right) \\
&\leq C(\Delta t)^2.
\end{aligned}$$

□

Corollary 3.3.6. *For all $1 \leq m \leq N$ the errors for the velocity can be bounded as*

$$\sqrt{\nu}\|\nabla\widetilde{\boldsymbol{\eta}}_u^m\|_0 + \sqrt{\gamma}\|\nabla\cdot\widetilde{\boldsymbol{\eta}}_u^m\|_0 \leq C\Delta t \tag{3.3.22}$$

Proof. Due to (3.3.21) we have:

$$\sum_{n=1}^{m}(\nu\|\nabla\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + \gamma\|\nabla\cdot\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2) \leq C(\Delta t)^3$$

and therefore via triangle inequality

$$\begin{aligned}
&\sqrt{\nu}\|\nabla\widetilde{\boldsymbol{\eta}}_u^m\|_0 + \sqrt{\gamma}\|\nabla\cdot\widetilde{\boldsymbol{\eta}}_u^m\|_0 \\
&\leq \sum_{n=1}^{m}(\sqrt{\nu}\|\nabla\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0 + \sqrt{\gamma}\|\nabla\cdot\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0) \\
&\leq C\left(m\sum_{n=1}^{m}(\nu\|\nabla\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + \gamma\|\nabla\cdot\delta_t\widetilde{\boldsymbol{\eta}}_u^n\|_0^2)\right)^{1/2} \leq C\Delta t.
\end{aligned}$$

□

The result so far gives us the correct rates of convergence with respect to the derivatives of the velocity. Lemma 3.3.7 aims to improve the estimate for the $L^2(\Omega)$-norm of the velocity.

Lemma 3.3.7. *For all $1 \leq m \leq N$ the velocity error can be bounded as*

$$\|\widetilde{\boldsymbol{\eta}}_u\|_{l^\infty(L^2(\Omega))}^2 \leq C\frac{(\Delta t)^4}{\nu^2}. \tag{3.3.23}$$

Proof. We eliminate $\boldsymbol{\eta}_u^n$ in the convection-diffusion error equation (3.3.15) using the projection error equation (3.3.17) and obtain

$$\begin{aligned}
&\left(\frac{3\widetilde{\boldsymbol{\eta}}_u^n - 4\widetilde{\boldsymbol{\eta}}_u^{n-1} + \widetilde{\boldsymbol{\eta}}_u^{n-2}}{2\Delta t}, \boldsymbol{v}_h\right) + \nu(\nabla\widetilde{\boldsymbol{\eta}}_u^n, \nabla\boldsymbol{v}_h) + \gamma(\nabla\cdot\widetilde{\boldsymbol{\xi}}^n, \nabla\cdot\boldsymbol{v}_h) - (\boldsymbol{R}^n, \boldsymbol{v}_h) \\
&= (\nabla(-p(t_n) + \frac{7}{3}r^{n-1} - \frac{5}{3}r^{n-2} + \frac{1}{3}r^{n-3}), \boldsymbol{v}_h) \\
&=: (\nabla\zeta^n, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h \cap \boldsymbol{Y}_h.
\end{aligned}$$

Now we test the equation with the inverse Stokes operator (A.4.1) applied to $4\Delta t\widetilde{\boldsymbol{\eta}}_u^n$:

$$\begin{aligned}
&(2(3\widetilde{\boldsymbol{\eta}}_u^n - 4\widetilde{\boldsymbol{\eta}}_u^{n-1} + \widetilde{\boldsymbol{\eta}}_u^{n-2}), S\widetilde{\boldsymbol{\eta}}_u^n) \\
&\quad + 4\Delta t\nu(\nabla\widetilde{\boldsymbol{\eta}}_u^n, \nabla S\widetilde{\boldsymbol{\eta}}_u^n) + 4\Delta t\gamma(\nabla\cdot\widetilde{\boldsymbol{\eta}}_u^n, \nabla\cdot S\widetilde{\boldsymbol{\eta}}_u^n) \\
&= 4\Delta t(\boldsymbol{R}^n, S\widetilde{\boldsymbol{\eta}}_u^n) + 4\Delta t(\nabla\zeta^n, S\widetilde{\boldsymbol{\eta}}_u^n) = 4\Delta t(\boldsymbol{R}^n, S\widetilde{\boldsymbol{\eta}}_u^n).
\end{aligned}$$

using $S\widetilde{\boldsymbol{\eta}}_u^n \in \boldsymbol{V}_h^{div}$. Recall that S is a self-adjoint operator:

$$\begin{aligned}(\boldsymbol{v}_h, S\boldsymbol{w}_h) &= \nu(\nabla S\boldsymbol{v}_h, \nabla S\boldsymbol{w}_h) + \gamma(\nabla\cdot S\boldsymbol{v}_h, \nabla\cdot S\boldsymbol{w}_h) + (r_{v_h}, S\boldsymbol{w}_h)\\ &= \nu(\nabla S\boldsymbol{w}_h, \nabla S\boldsymbol{v}_h) + \gamma(\nabla\cdot S\boldsymbol{w}_h, \nabla\cdot S\boldsymbol{v}_h) + (r_{w_h}, S\boldsymbol{w}_h)\\ &= (\boldsymbol{v}_h, S\boldsymbol{w}_h).\end{aligned}$$

By the definition of the induced semi-norm $|\boldsymbol{u}|_* = (\boldsymbol{u}, S\boldsymbol{u})$ and the splitting (A.1.1) we get

$$\begin{aligned}&|\widetilde{\boldsymbol{\eta}}_u^n|_*^2 + |2\widetilde{\boldsymbol{\eta}}_u^n - \widetilde{\boldsymbol{\eta}}_u^{n-1}|_*^2 + |\delta_{tt}\widetilde{\boldsymbol{\eta}}_u^n|_*^2\\ &\quad + 4\Delta t\nu(\nabla\widetilde{\boldsymbol{\eta}}_u^n, \nabla S\widetilde{\boldsymbol{\eta}}_u^n) + 4\Delta t\gamma(\nabla\cdot\widetilde{\boldsymbol{\eta}}_u^n, \nabla\cdot S\widetilde{\boldsymbol{\eta}}_u^n)\\ &= 4\Delta t(\boldsymbol{R}^n, S\widetilde{\boldsymbol{\eta}}_u^n) + |\widetilde{\boldsymbol{\eta}}_u^{n-1}|_*^2 + |2\widetilde{\boldsymbol{\eta}}_u^{n-1} - \widetilde{\boldsymbol{\eta}}_u^{n-2}|_*^2.\end{aligned} \tag{3.3.24}$$

The consistency error can be bounded as

$$\begin{aligned}4\Delta t(\boldsymbol{R}^n, S\widetilde{\boldsymbol{\eta}}_u^n) &\le 4\frac{\Delta t}{\nu}\|\boldsymbol{R}^n\|_{-1}^2 + \nu\Delta t\|S\widetilde{\boldsymbol{\eta}}_u^n\|_1^2 \overset{(A.4.5)}{\le} 4\frac{\Delta t}{\nu}\|\boldsymbol{R}^n\|_{-1}^2 + \Delta t\|\widetilde{\boldsymbol{\eta}}_u^n\|_{-1}^2\\ &\le 4\frac{\Delta t}{\nu}\|\boldsymbol{R}^n\|_{-1}^2 + \Delta t\|\widetilde{\boldsymbol{\eta}}_u^n\|_0^2.\end{aligned}$$

Using A.4.7 with $\epsilon = 2\left(\frac{2\nu+\gamma}{\nu}\right)^{-2}$, the diffusive term and the grad-div stabilization can be estimated by

$$4\Delta t\nu(\nabla\widetilde{\boldsymbol{\eta}}_u^n, \nabla S\widetilde{\boldsymbol{\eta}}_u^n) + 4\Delta t\gamma(\nabla\cdot\widetilde{\boldsymbol{\eta}}_u^n, \nabla\cdot S\widetilde{\boldsymbol{\eta}}_u^n) \ge 2\Delta t\|\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 - c\Delta t\|\widetilde{\boldsymbol{\eta}}_u^n - \boldsymbol{\eta}_u^n\|_0^2.$$

where $c = 2\left(\frac{2\nu+\gamma}{\nu}\right)^2$. Combining these estimates we arrive at

$$\begin{aligned}&|\widetilde{\boldsymbol{\eta}}_u^n|_*^2 + |2\widetilde{\boldsymbol{\eta}}_u^n - \widetilde{\boldsymbol{\eta}}_u^{n-1}|_*^2 + |\delta_{tt}\widetilde{\boldsymbol{\eta}}_u^n|_*^2 + \Delta t\|\widetilde{\boldsymbol{\eta}}_u^n\|_0^2\\ &\le 4\frac{\Delta t}{\nu}\|\boldsymbol{R}^n\|_{-1}^2 + |\widetilde{\boldsymbol{\eta}}_u^{n-1}|_*^2 + |2\widetilde{\boldsymbol{\eta}}_u^{n-1} - \widetilde{\boldsymbol{\eta}}_u^{n-2}|_*^2 + c\Delta t\|\widetilde{\boldsymbol{\eta}}_u^n - \boldsymbol{\eta}_u^n\|_0^2\end{aligned}$$

that yields summed up

$$\begin{aligned}&|\widetilde{\boldsymbol{\eta}}_u^m|_*^2 + |2\widetilde{\boldsymbol{\eta}}_u^m - \widetilde{\boldsymbol{\eta}}_u^{m-1}|_*^2 + \sum_{n=3}^m(|\delta_{tt}\widetilde{\boldsymbol{\eta}}_u^n|_*^2 + \Delta t\|\widetilde{\boldsymbol{\eta}}_u^n\|_0^2)\\ &\le |\widetilde{\boldsymbol{\eta}}_u^2|_*^2 + |2\widetilde{\boldsymbol{\eta}}_u^2 - \widetilde{\boldsymbol{\eta}}_u^1|_*^2 + \sum_{n=3}^m\left(4\frac{\Delta t}{\nu}\|\boldsymbol{R}^n\|_{-1}^2 + c\Delta t\|\widetilde{\boldsymbol{\eta}}_u^n - \boldsymbol{\eta}_u^n\|_0^2\right)\\ &\overset{(A.4.4)}{\le} \|\widetilde{\boldsymbol{\eta}}_u^2\|_0^2 + \|2\widetilde{\boldsymbol{\eta}}_u^2 - \widetilde{\boldsymbol{\eta}}_u^1\|_0^2 + C\Delta t\sum_{n=3}^m\left(\frac{\|\boldsymbol{R}^n\|_{-1}^2}{\nu} + \|\widetilde{\boldsymbol{\eta}}_u^n - \boldsymbol{\eta}_u^n\|_0^2\right)\\ &\le C\frac{(\Delta t)^4}{\nu}\end{aligned} \tag{3.3.25}$$

due to the initial error estimates. In particular, we derive

$$\|\widetilde{\boldsymbol{\eta}}_u\|_{l^2(t_0,T;[L^2(\Omega)]^d)}^2 = \Delta t\sum_{n=0}^N\|\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 \le C\frac{(\Delta t)^4}{\nu}. \tag{3.3.26}$$

For an l^∞-estimate we use (A.4.7) again with $\epsilon = 2\left(\frac{\nu}{2\nu+\gamma}\right)^2$ and obtain from Lemma (3.3.5) together with (3.3.25)

$$\begin{aligned}
\|\widetilde{\boldsymbol{\eta}}_u\|^2_{l^\infty(t_0,T;[L^2(\Omega)]^d)} &\leq C \max_{1\leq n\leq N} |\widetilde{\boldsymbol{\eta}}^n_u|^2_* + C\left(1+\frac{\gamma}{\nu}\right)^2 \max_{1\leq n\leq N} \|\widetilde{\boldsymbol{\eta}}^n_u - \boldsymbol{\eta}^n_u\|^2_0 \\
&\leq C\frac{(\Delta t)^4}{\nu^2}.
\end{aligned}$$

□

Now we proved everything we wanted with respect to the velocity for the nonlinear auxiliary problem. For the pressure we do not give a convergence result at this point but just at the end of the chapter.

3.3.4 Error Estimates for the Nonlinear Problem

For the nonlinear error we get a superconvergent result with respect to the derivatives of the velocity.

Lemma 3.3.8. *For all $1 \leq m \leq N$ it holds*

$$\begin{aligned}
&\|\widetilde{\boldsymbol{e}}^m_u\|^2_0 + (\Delta t)^2\|\nabla e^m_p\|^2_0 + \sum_{n=1}^m \Big[\Delta t\nu\|\nabla\widetilde{\boldsymbol{e}}^n_u\|^2_0 + \Delta t\gamma\|\nabla\cdot\widetilde{\boldsymbol{e}}^n_u\|^2_0 \\
&\quad + \Delta t \sum_{M\in\mathcal{M}_h} \tau^n_M \|\kappa_M((\widetilde{\boldsymbol{u}}^n_M\cdot\nabla)\widetilde{\boldsymbol{e}}^n_u)\|^2_{0,M}\Big] \\
&\leq C_{G,t}\Bigg(\frac{(\Delta t)^4}{\nu^3 h^{2d}} \max_{1\leq n\leq m}\max_{M\in\mathcal{M}_h}\{\tau^n_M\}^2 + \frac{(\Delta t)^4}{\nu^2 h^d}\max_{1\leq n\leq m}\max_{M\in\mathcal{M}_h}\{\tau^n_M\} \\
&\qquad + \frac{(\Delta t)^2}{\nu}\max_{1\leq n\leq m}\max_{M\in\mathcal{M}_h}\{\tau^n_M|\widetilde{\boldsymbol{u}}^n_M|^2\} \\
&\qquad + \frac{\Delta t h^{2k_u}}{\nu^3} + \frac{(\Delta t)^{2l+1}}{\nu^2} + \frac{(\Delta t)^4}{\nu^3}\Bigg)
\end{aligned} \tag{3.3.27}$$

where the Gronwall term $C_{G,t}$ is defined by

$$\begin{aligned}
C_{G,t} &\sim \exp\left(\frac{T}{1-K}\right) \\
K &:= C\Delta t\Bigg(\frac{h^{4k_u}+\nu^2(\Delta t)^{4l}}{(\Delta t)^2\nu^5} + \frac{1}{\nu} \\
&\qquad + \max_{1\leq n\leq m}\max_{M\in\mathcal{M}_h}\left\{\frac{(\tau^n_M)^2}{\nu h^{2d}}\right\} + \max_{1\leq n\leq m}\max_{M\in\mathcal{M}_h}\left\{\frac{\tau^n_M}{h^d}\right\}\Bigg)
\end{aligned}$$

provided that $K < 1$.

Proof. Subtracting the convection-diffusion and projection equations for $\widetilde{\boldsymbol{w}}_{ht}^n$ and $\widetilde{\boldsymbol{u}}_{ht}^n$ from each other gives

$$\begin{aligned}\left(\frac{3\widetilde{\boldsymbol{e}}_u^n - 4\boldsymbol{e}_u^{n-1} + \boldsymbol{e}_u^{n-2}}{2\Delta t}, \boldsymbol{v}_h\right) + \nu(\nabla\widetilde{\boldsymbol{e}}_u^n, \nabla\boldsymbol{v}_h) + \gamma(\nabla\cdot\widetilde{\boldsymbol{e}}_u^n, \nabla\cdot\boldsymbol{v}_h) \\ + (\nabla e_p^{n-1}, \boldsymbol{v}_h) + c(\boldsymbol{u}_h(t_n), \boldsymbol{u}_h(t_n), \boldsymbol{v}_h) - c(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) \\ + s_h(\boldsymbol{u}_h(t_n), \boldsymbol{u}_h(t_n), \boldsymbol{u}_h(t_n), \boldsymbol{v}_h) - s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) = 0\end{aligned} \tag{3.3.28}$$

and

$$\left(\frac{3\boldsymbol{e}_u^n - 3\widetilde{\boldsymbol{e}}_u^n}{2\Delta t} + \nabla(e_p^n - e_p^{n-1}), \boldsymbol{y}_h\right) = 0. \tag{3.3.29}$$

We now test (3.3.28) with $4\Delta t\boldsymbol{e}_u^n$ and (3.3.29) with $\Delta t\,\nabla e_p^{n-1}$:

$$\begin{aligned}&2\left(3\widetilde{\boldsymbol{e}}_u^n - 4\boldsymbol{e}_u^{n-1} + \boldsymbol{e}_u^{n-2}, \widetilde{\boldsymbol{e}}_u^n\right) \\ &\quad + 4\Delta t\nu(\nabla\widetilde{\boldsymbol{e}}_u^n, \nabla\widetilde{\boldsymbol{e}}_u^n) + 4\Delta t\gamma(\nabla\cdot\widetilde{\boldsymbol{e}}_u^n, \nabla\cdot\widetilde{\boldsymbol{e}}_u^n) \\ &\quad + 4\Delta t\left(s_h(\boldsymbol{u}_h(t_n), \boldsymbol{u}_h(t_n), \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{e}}_u^n) - s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{e}}_u^n)\right) \\ &= -4\Delta t\left(c(\boldsymbol{u}_h(t_n), \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{e}}_u^n) - c(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{e}}_u^n)\right) - 4\Delta t(\nabla e_p^{n-1}, \widetilde{\boldsymbol{e}}_u^n)\end{aligned} \tag{3.3.30}$$

$$\left(\frac{3}{2\Delta t}\widetilde{\boldsymbol{e}}_u^n, \nabla e_p^{n-1}\right) = (\nabla(e_p^n - e_p^{n-1}), \nabla e_p^{n-1}) \tag{3.3.31}$$

We add these equations using the usual splitting (A.1.1) for the terms that resemble the discretized time derivative

$$\begin{aligned}&(2(3\widetilde{\boldsymbol{e}}_u^n - 4\boldsymbol{e}_u^{n-1} + \boldsymbol{e}_u^{n-2}), \widetilde{\boldsymbol{e}}_u^n) = I_1 + I_2 + I_3 \\ &= 3\|\widetilde{\boldsymbol{e}}_u^n\|_0^2 + 3\|\boldsymbol{e}_u^n - \widetilde{\boldsymbol{e}_u}^n\|_0^2 - 2\|\boldsymbol{e}_u^n\|_0^2 + \|2\boldsymbol{e}_u^n - \boldsymbol{e}_u^{n-1}\|_0^2 \\ &\quad + \|\delta_{tt}\boldsymbol{e}_u^n\|_0^2 - \|\boldsymbol{e}_u^{n-1}\|_0^2 - \|2\boldsymbol{e}_u^{n-1} - \boldsymbol{e}_u^{n-2}\|_0^2.\end{aligned}$$

The second term I_2 again vanishes due to the fact that $\boldsymbol{e}_u^n$ is weakly divergence-free. With the identity $(a-b, b) = \frac{1}{2}(\|a\|_0^2 - \|a-b\|_0^2 - \|b\|_0^2)$ and $\|\boldsymbol{e}_u^n\| \overset{1.2.1}{\leq} \|\widetilde{\boldsymbol{e}}_u^n\|$ we have

$$\begin{aligned}&\|\widetilde{\boldsymbol{e}}_u^n\|_0^2 + 3\|\boldsymbol{e}_u^n - \widetilde{\boldsymbol{e}}_u^n\|_0^2 + \|2\boldsymbol{e}_u^n - \boldsymbol{e}_u^{n-1}\|_0^2 + \frac{4}{3}(\Delta t)^2\|\nabla e_p^n\|_0^2 \\ &\quad + 4\Delta t\nu\|\nabla\widetilde{\boldsymbol{e}}_u^n\|_0^2 + 4\Delta t\gamma\|\nabla\cdot\widetilde{\boldsymbol{e}}_u^n\|_0^2 + \|\delta_{tt}\boldsymbol{e}_u^n\|_0^2 + 4\Delta t(\boldsymbol{S}^n, \widetilde{\boldsymbol{e}}_u^n) \\ &\leq \|\widetilde{\boldsymbol{e}}_u^{n-1}\|_0^2 + \|2\boldsymbol{e}_u^{n-1} - \boldsymbol{e}_u^{n-2}\|_0^2 + \frac{4}{3}(\Delta t)^2\|\nabla e_p^{n-1}\|_0^2 \\ &\quad + \frac{4}{3}(\Delta t)^2\|\nabla(e_p^n - e_p^{n-1})\|_0^2 - 4\Delta t(\boldsymbol{Q}^n, \widetilde{\boldsymbol{e}}_u^n)\end{aligned} \tag{3.3.32}$$

where $\boldsymbol{Q}^n$ abbreviates the convective terms and $\boldsymbol{S}^n$ the terms due to SU stabilization:

$$(\boldsymbol{Q}^n, \widetilde{\boldsymbol{e}}_u^n) := c(\boldsymbol{u}_h(t_n), \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{e}}_u^n) - c(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{e}}_u^n)$$

$$(\boldsymbol{S}^n, \widetilde{\boldsymbol{e}}_u^n) := s_h(\boldsymbol{u}_h(t_n), \boldsymbol{u}_h(t_n), \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{e}}_u^n) - s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{e}}_u^n)$$

The projection equation (3.3.29) tested with $\nabla(e_p^n - e_p^{n-1})$ yields

$$\frac{4}{3}(\Delta t)^2 \|\nabla(e_p^n - e_p^{n-1})\|_0^2 \leq 3\|\boldsymbol{e}_u^n - \widetilde{\boldsymbol{e}}_u^n\|_0^2 \tag{3.3.33}$$

and thus after summation from $n = 2$ to m:

$$\begin{aligned} &\|\widetilde{\boldsymbol{e}}_u^m\|_0^2 + \|2\boldsymbol{e}_u^m - \boldsymbol{e}_u^{m-1}\|_0^2 + \frac{4}{3}(\Delta t)^2 \|\nabla e_p^m\|_0^2 \\ &\quad + \sum_{n=2}^{m} \Big[4\Delta t \nu \|\nabla \widetilde{\boldsymbol{e}}_u^n\|_0^2 + 4\Delta t \gamma \|\nabla \cdot \widetilde{\boldsymbol{e}}_u^n\|_0^2 + \|\delta_{tt} \boldsymbol{e}_u^n\|_0^2 + 4\Delta t (\boldsymbol{S}^n, \widetilde{\boldsymbol{e}}_u^n) \Big] \\ &\leq \widetilde{\boldsymbol{e}}_u^1\|_0^2 + \|2\boldsymbol{e}_u^1\|_0^2 + \frac{4}{3}(\Delta t)^2 \|\nabla e_p^1\|_0^2 - \sum_{n=2}^{m} 4\Delta t (\boldsymbol{Q}^n, \widetilde{\boldsymbol{e}}_u^n) \end{aligned} \tag{3.3.34}$$

Due to $\widetilde{\boldsymbol{e}}_u^n + \widetilde{\boldsymbol{\eta}}_u^n = \boldsymbol{u}_h(t_n) - \widetilde{\boldsymbol{u}}_{ht}^n$ we calculate for the convective term using skew-symmetry

$$\begin{aligned} (\boldsymbol{Q}^n, \widetilde{\boldsymbol{e}}_u^n) &= c(\boldsymbol{u}_h(t_n), \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{e}}_u^n) - c(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{e}}_u^n) \\ &= c(\widetilde{\boldsymbol{\eta}}_u^n + \widetilde{\boldsymbol{e}}_u^n, \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{e}}_u^n) + c(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{\eta}}_u^n + \widetilde{\boldsymbol{e}}_u^n, \widetilde{\boldsymbol{e}}_u^n) \\ &= c(\widetilde{\boldsymbol{\eta}}_u^n + \widetilde{\boldsymbol{e}}_u^n, \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{e}}_u^n) + c(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{e}}_u^n) \\ &= c(\widetilde{\boldsymbol{e}}_u^n, \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{e}}_u^n) + c(\widetilde{\boldsymbol{\eta}}_u^n, \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{e}}_u^n) + c(\boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{e}}_u^n) \\ &\quad - c(\widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{e}}_u^n) - c(\widetilde{\boldsymbol{e}}_u^n, \widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{e}}_u^n) \end{aligned} \tag{3.3.35}$$

and with boundedness of $\|\nabla \boldsymbol{u}_h\|_0$ (cf. 3.2.4):

$$\begin{aligned} &c(\widetilde{\boldsymbol{e}}_u^n, \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{e}}_u^n) = c(\widetilde{\boldsymbol{e}}_u^n, \boldsymbol{u}(t_n), \widetilde{\boldsymbol{e}}_u^n) + c(\widetilde{\boldsymbol{e}}_u^n, \boldsymbol{u}_h(t_n) - \boldsymbol{u}(t_n), \widetilde{\boldsymbol{e}}_u^n) \\ &\quad \leq C\|\widetilde{\boldsymbol{e}}_u^n\|_0 \|\boldsymbol{u}(t_n)\|_2 \|\widetilde{\boldsymbol{e}}_u^n\|_1 + C\|\widetilde{\boldsymbol{e}}_u^n\|_0^{1/2} \|\boldsymbol{u}_h(t_n) - \boldsymbol{u}_h(t_n)\|_1 \|\widetilde{\boldsymbol{e}}_u^n\|_1^{3/2} \\ &\quad \leq \frac{\nu}{32}\|\nabla \widetilde{\boldsymbol{e}}_u^n\|_0^2 + C\left(\frac{\|\boldsymbol{u}_h(t_n) - \boldsymbol{u}_h(t_n)\|_1^4}{\nu^3} + \frac{1}{\nu} \right) \|\widetilde{\boldsymbol{e}}_u^n\|_0^2 \\ &c(\boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{e}}_u^n) + c(\widetilde{\boldsymbol{\eta}}_u^n, \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{e}}_u^n) \\ &\quad = c(\boldsymbol{u}(t_n), \widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{e}}_u^n) - c(\boldsymbol{u}(t_n) - \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{e}}_u^n) \\ &\qquad + c(\widetilde{\boldsymbol{\eta}}_u^n, \boldsymbol{u}(t_n), \widetilde{\boldsymbol{e}}_u^n) - c(\widetilde{\boldsymbol{\eta}}_u^n, \boldsymbol{u}(t_n) - \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{e}}_u^n) \\ &\quad \leq C\|\boldsymbol{u}(t_n)\|_2 \|\widetilde{\boldsymbol{\eta}}_u^n\|_0 \|\widetilde{\boldsymbol{e}}_u^n\|_1 + C\|\boldsymbol{u}(t_n) - \boldsymbol{u}_h(t_n)\|_1 \|\widetilde{\boldsymbol{\eta}}_u^n\|_1 \|\widetilde{\boldsymbol{e}}_u^n\|_1 \\ &\quad \leq \frac{\nu}{32}\|\nabla \widetilde{\boldsymbol{e}}_u^n\|_0^2 + \frac{C}{\nu}\|\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + \frac{C}{\nu}\|\widetilde{\boldsymbol{\eta}}_u^n\|_1^2 \|\boldsymbol{u}(t_n) - \boldsymbol{u}_h(t_n)\|_1^2 \\ &c(\widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{e}}_u^n) \leq C\|\widetilde{\boldsymbol{\eta}}_u^n\|_1^2 \|\widetilde{\boldsymbol{e}}_u^n\|_1 \leq \frac{\nu}{32}\|\nabla \widetilde{\boldsymbol{e}}_u^n\|_0^2 + \frac{C}{\nu}\|\widetilde{\boldsymbol{\eta}}_u^n\|_1^4 \\ &c(\widetilde{\boldsymbol{e}}_u^n, \widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{e}}_u^n) \leq C\|\widetilde{\boldsymbol{\eta}}_u^n\|_1 \|\widetilde{\boldsymbol{e}}_u^n\|_1^2 \end{aligned} \tag{3.3.36}$$

Using $\|\widetilde{\boldsymbol{\eta}}_u^n\|_0 \leq C\frac{(\Delta t)^2}{\nu}$, the estimate (3.2.4) for $\|\boldsymbol{u}(t_n) - \boldsymbol{u}_h(t_n)\|_1^2$ and $\|\widetilde{\boldsymbol{\eta}}_u^n\|_1 \leq C\Delta t \leq C\nu$ we get in combination

$$
\begin{aligned}
(\boldsymbol{Q}^n, \widetilde{\boldsymbol{e}}_u^n) \leq{}& \frac{\nu}{8}\|\nabla\widetilde{\boldsymbol{e}}_u^n\|_0^2 + C\left(\frac{h^{4k_u}+\nu^2(\Delta t)^{4l}}{(\Delta t)^2\nu^5} + \frac{1}{\nu}\right)\|\widetilde{\boldsymbol{e}}_u^n\|_0^2 \\
&+ C\frac{\Delta t h^{2k_u}}{\nu^3} + C\frac{(\Delta t)^{2l+1}}{\nu^2} + C\frac{(\Delta t)^4}{\nu^3}.
\end{aligned}
$$

For the stabilization term we have

$$
\begin{aligned}
(\boldsymbol{S}^n, \widetilde{\boldsymbol{e}}_u^n) &= s_h(\boldsymbol{u}_h(t_n), \boldsymbol{u}_h(t_n), \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{e}}_u^n) - s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{e}}_u^n) \\
&= s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{\eta}}_u^n + \widetilde{\boldsymbol{e}}_u^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{e}}_u^n) + s_h(\widetilde{\boldsymbol{\eta}}_u^n + \widetilde{\boldsymbol{e}}_u^n, \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{e}}_u^n) \\
&\quad + s_h(\boldsymbol{u}_h(t_n), \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{\eta}}_u^n + \widetilde{\boldsymbol{e}}_u^n, \widetilde{\boldsymbol{e}}_u^n) \\
&= s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{e}}_u^n) + s_h(\widetilde{\boldsymbol{\eta}}_u^n, \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{e}}_u^n) \\
&\quad + s_h(\boldsymbol{u}_h(t_n), \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{e}}_u^n) + s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{e}}_u^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{e}}_u^n) \\
&\quad + s_h(\widetilde{\boldsymbol{e}}_u^n, \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{e}}_u^n) + s_h(\boldsymbol{u}_h(t_n), \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{e}}_u^n, \widetilde{\boldsymbol{e}}_u^n) \\
&=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6
\end{aligned}
\tag{3.3.37}
$$

The terms $I_5 + I_6$ can be handled as

$$
\begin{aligned}
|I_5 + I_6| \leq{}& \frac{C}{h^d}\max_{M\in\mathcal{M}_h}\{\tau_M^n\}\|\widetilde{\boldsymbol{e}}_u^n\|_0^2 + \frac{1}{4}\sum_{M\in\mathcal{M}_h}\tau_M^n\|\kappa_M((\widetilde{\boldsymbol{u}}_M^n\cdot\nabla)\widetilde{\boldsymbol{e}}_u^n)\|_{0,M}^2 \\
&+ \frac{\nu}{8}\|\widetilde{\boldsymbol{e}}_u^n\|_1^2 + \frac{C\max_{M\in\mathcal{M}_h}\{\tau_M^n\}^2}{\nu h^{2d}}\|\widetilde{\boldsymbol{e}}_u^n\|_0^2
\end{aligned}
$$

and we have for the remaining terms:

$$
\begin{aligned}
I_1 &= s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{e}}_u^n) \\
&\leq C\sqrt{\max_{M\in\mathcal{M}_h}\{\tau_M^n|\widetilde{\boldsymbol{u}}_M^n|^2\}}\|\widetilde{\boldsymbol{\eta}}_u^n\|_1\left(\sum_{M\in\mathcal{M}_h}\tau_M^n\|\kappa_M((\widetilde{\boldsymbol{u}}_M^n\cdot\nabla)\widetilde{\boldsymbol{e}}_u^n)\|_{0,M}^2\right)^{1/2} \\
&\leq C\max_{M\in\mathcal{M}_h}\{\tau_M^n|\widetilde{\boldsymbol{u}}_M^n|^2\}\|\widetilde{\boldsymbol{\eta}}_u^n\|_1^2 + \frac{1}{4}\sum_{M\in\mathcal{M}_h}\tau_M^n\|\kappa_M((\widetilde{\boldsymbol{u}}_M^n\cdot\nabla)\widetilde{\boldsymbol{e}}_u^n)\|_{0,M}^2 \\
&\leq \frac{C}{\nu}\max_{M\in\mathcal{M}_h}\{\tau_M^n|\widetilde{\boldsymbol{u}}_M^n|^2\}(\Delta t)^2 + \frac{1}{4}\sum_{M\in\mathcal{M}_h}\tau_M^n\|\kappa_M((\widetilde{\boldsymbol{u}}_M^n\cdot\nabla)\widetilde{\boldsymbol{e}}_u^n)\|_{0,M}^2, \\
I_2 &= s_h(\widetilde{\boldsymbol{\eta}}_u^n, \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{e}}_u^n) \\
&\leq \frac{C}{h^d}\max_{M\in\mathcal{M}_h}\{\tau_M^n\}\|\widetilde{\boldsymbol{\eta}}_u^n\|_0^2\|\boldsymbol{u}_h(t_n)\|_1^2 + \frac{1}{4}\sum_{M\in\mathcal{M}_h}\tau_M^n\|\kappa_M((\widetilde{\boldsymbol{u}}_M^n\cdot\nabla)\widetilde{\boldsymbol{e}}_u^n)\|_{0,M}^2 \\
&\leq \frac{C}{h^d\nu^2}\max_{M\in\mathcal{M}_h}\{\tau_M^n\}(\Delta t)^4 + \frac{1}{4}\sum_{M\in\mathcal{M}_h}\tau_M^n\|\kappa_M((\widetilde{\boldsymbol{u}}_M^n\cdot\nabla)\widetilde{\boldsymbol{e}}_u^n)\|_{0,M}^2, \\
I_3 &= s_h(\boldsymbol{u}_h(t_n), \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{e}}_u^n) \\
&\leq \frac{C}{\nu h^{2d}}\max_{M\in\mathcal{M}_h}\{\tau_M^n\}^2\|\boldsymbol{u}_h(t_n)\|_0^2\|\boldsymbol{u}_h(t_n)\|_1^2\|\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + \frac{\nu}{4}\|\widetilde{\boldsymbol{e}}_u^n\|_1^2
\end{aligned}
$$

$$\le \frac{C}{\nu^3 h^{2d}} \max_{M\in\mathcal{M}_h} \{\tau_M^n\}^2 (\Delta t)^4 + \frac{\nu}{4}\|\widetilde{\boldsymbol{e}}_u^n\|_1^2.$$

Taking these terms together yields

$$\begin{aligned}
-(S^n, \widetilde{\boldsymbol{e}}_u^n) + I_4 \le \frac{3}{4} \sum_{M\in\mathcal{M}_h} \tau_M^n \|\kappa_M((\widetilde{\boldsymbol{u}}_M^n \cdot \nabla)\widetilde{\boldsymbol{e}}_u^n)\|_{0,M}^2 + \frac{3\nu}{8}\|\widetilde{\boldsymbol{e}}_u^n\|_1^2 \\
+ \left(\frac{C \max_{M\in\mathcal{M}_h}\{\tau_M^n\}}{h^d} + \frac{C \max_{M\in\mathcal{M}_h}\{\tau_M^n\}^2}{\nu h^{2d}} \right) \|\widetilde{\boldsymbol{e}}_u^n\|_0^2 \\
+ \left(\frac{C \max_{M\in\mathcal{M}_h}\{\tau_M^n\}}{\nu^2 h^d} + \frac{C \max_{M\in\mathcal{M}_h}\{\tau_M^n\}^2}{\nu^3 h^{2d}} \right) (\Delta t)^4 \\
+ \frac{C}{\nu} \max_{M\in\mathcal{M}_h} \{\tau_M^n |\widetilde{\boldsymbol{u}}_M^n|^2\} (\Delta t)^2.
\end{aligned}$$

Inserting all these estimates into (3.3.34) gives

$$\begin{aligned}
&\|\widetilde{\boldsymbol{e}}_u^m\|_0^2 + \|2\boldsymbol{e}_u^m - \boldsymbol{e}_u^{m-1}\|_0^2 + \frac{4}{3}(\Delta t)^2 \|\nabla e_p^m\|_0^2 \\
&\quad + \sum_{n=2}^{m} \Big[4\Delta t \nu \|\nabla \widetilde{\boldsymbol{e}}_u^n\|_0^2 + 4\Delta t \gamma \|\nabla\cdot \widetilde{\boldsymbol{e}}_u^n\|_0^2 + \|\delta_{tt}\boldsymbol{e}_u^n\|_0^2 \\
&\qquad + \Delta t \sum_{M\in\mathcal{M}_h} \tau_M^n \|\kappa_M((\widetilde{\boldsymbol{u}}_M^n\cdot\nabla)\widetilde{\boldsymbol{e}}_u^n)\|_{0,M}^2 \Big] \\
&\le \|\widetilde{\boldsymbol{e}}_u^1\|_0^2 + \|2\boldsymbol{e}_u^1 - \boldsymbol{e}_u^0\|_0^2 + \frac{4}{3}(\Delta t)^2 \|\nabla e_p^1\|_0^2 \\
&\quad + 4\Delta t \sum_{n=2}^{m} \Bigg(\Bigg(C\frac{h^{4k_u} + \nu^2 (\Delta t)^{4l}}{(\Delta t)^2 \nu^5} + \frac{C}{\nu} \\
&\qquad + \frac{C}{h^d} \max_{M\in\mathcal{M}_h}\{\tau_M^n\} + \frac{C \max_{M\in\mathcal{M}_h}\{\tau_M^n\}^2}{\nu h^{2d}} \Bigg) \|\widetilde{\boldsymbol{e}}_u^n\|_0^2 \qquad (3.3.38) \\
&\qquad + C \Bigg(\frac{(\Delta t)^4 \max_{M\in\mathcal{M}_h}\{\tau_M^n\}^2}{\nu h^{2d}} \\
&\qquad\quad + \frac{(\Delta t)^4 \max_{M\in\mathcal{M}_h}\{\tau_M^n\}}{\nu h^d} \\
&\qquad\quad + (\Delta t)^2 \max_{M\in\mathcal{M}_h}\{\tau_M^n |\widetilde{\boldsymbol{u}}_M^n|^2\} \Bigg) + \frac{\nu}{2}\|\nabla\widetilde{\boldsymbol{e}}_u^n\|_0^2 \\
&\qquad + C\Delta t h^{2k_u} + C(\Delta t)^{2l+1} + C\frac{(\Delta t)^4}{\nu^3} \\
&\qquad + \frac{3}{4} \sum_{M\in\mathcal{M}_h} \tau_M^n \|\kappa_M((\widetilde{\boldsymbol{u}}_M^n\cdot\nabla)\widetilde{\boldsymbol{e}}_u^n)\|_{0,M}^2 \Bigg)
\end{aligned}$$

and therefore

$$
\begin{aligned}
&\|\widetilde{\boldsymbol{e}}_u^m\|_0^2 + \|2\boldsymbol{e}_u^m - \boldsymbol{e}_u^{m-1}\|_0^2 + \frac{4}{3}(\Delta t)^2 \|\nabla e_p^m\|_0^2 \\
&\quad + \sum_{n=2}^{m} \Big[2\Delta t \nu \|\nabla \widetilde{\boldsymbol{e}}_u^n\|_0^2 + 4\Delta t \gamma \|\nabla \cdot \widetilde{\boldsymbol{e}}_u^n\|_0^2 + \|\delta_{tt} \boldsymbol{e}_u^n\|_0^2 \\
&\qquad\quad + \Delta t \sum_{M \in \mathcal{M}_h} \tau_M^n \|\kappa_M((\widetilde{\boldsymbol{u}}_M^n \cdot \nabla)\widetilde{\boldsymbol{e}}_u^n)\|_{0,M}^2 \Big] \\
&\leq \|\widetilde{\boldsymbol{e}}_u^1\|_0^2 + \|2\boldsymbol{e}_u^1 - \boldsymbol{e}_u^0\|_0^2 + \frac{4}{3}(\Delta t)^2 \|\nabla e_p^1\|_0^2 \\
&\quad + C\Delta t \sum_{n=2}^{m} \left(\frac{h^{4k_u} + \nu^2 (\Delta t)^{4l}}{(\Delta t)^2 \nu^5} + \frac{1}{\nu} \right. \\
&\qquad\qquad \left. + \frac{\max_{M \in \mathcal{M}_h} \{\tau_M^n\}}{h^d} + \frac{\max_{M \in \mathcal{M}_h} \{\tau_M^n\}^2}{\nu h^{2d}} \right) \|\widetilde{\boldsymbol{e}}_u^n\|_0^2 \\
&\quad + C(\Delta t)^2 \left(\frac{(\Delta t)^2 \max_{M \in \mathcal{M}_h} \max_{1 \leq n \leq m} \{\tau_M^n\}^2}{\nu h^{2d}} \right. \\
&\qquad\qquad + \frac{(\Delta t)^2 \max_{M \in \mathcal{M}_h} \max_{1 \leq n \leq m} \{\tau_M^n\}}{\nu h^d} \\
&\qquad\qquad \left. + \max_{M \in \mathcal{M}_h} \{\tau_M^n |\widetilde{\boldsymbol{u}}_M^n|^2\} \right) \\
&\quad + C\Delta t h^{2k_u} + C(\Delta t)^{2l+1} + C\frac{(\Delta t)^4}{\nu^3}.
\end{aligned}
\tag{3.3.39}
$$

Due to the estimates for the initial errors of the time discretized problem and the linear auxiliary problem, the initial errors also converge suitably

$$
\|\widetilde{\boldsymbol{e}}_u^1\|_0^2 + \|2\boldsymbol{e}_u^1 - \boldsymbol{e}_u^0\|_0^2 + \frac{4}{3}(\Delta t)^2 \|\nabla e_p^1\|_0^2 \leq C(\Delta t)^4.
$$

Now we can use the discrete Gronwall Lemma in (3.3.39) for $\widetilde{\boldsymbol{e}}_u$ and obtain

$$
\begin{aligned}
&\|\widetilde{\boldsymbol{e}}_u^m\|_0^2 + (\Delta t)^2 \|\nabla e_p^m\|_0^2 \\
&\quad + \Delta t \sum_{n=1}^{m} \Big[\nu \|\nabla \widetilde{\boldsymbol{e}}_u^n\|_0^2 + \gamma \|\nabla \cdot \widetilde{\boldsymbol{e}}_u^n\|_0^2 \\
&\qquad\qquad + \sum_{M\in\mathcal{M}_h} \tau_M^n \|\kappa_M((\widetilde{\boldsymbol{u}}_M^n \cdot \nabla)\widetilde{\boldsymbol{e}}_u^n)\|_{0,M}^2 \Big] \\
&\le C_{G,t} \Bigg(\frac{(\Delta t)^4 \max_{1\le n\le m} \max_{M\in\mathcal{M}_h} \{\tau_M^n\}^2}{\nu^3 h^{2d}} \\
&\qquad + \frac{(\Delta t)^4 \max_{1\le n\le m} \max_{M\in\mathcal{M}_h} \{\tau_M^n\}}{\nu^2 h^d} \\
&\qquad + \frac{(\Delta t)^2}{\nu} \max_{1\le n\le m} \max_{M\in\mathcal{M}_h} \{\tau_M^n |\widetilde{\boldsymbol{u}}_M^n|^2\} \\
&\qquad + \frac{\Delta t h^{2k_u}}{\nu^3} + \frac{(\Delta t)^{2l+1}}{\nu^2} + \frac{(\Delta t)^4}{\nu^3} \Bigg)
\end{aligned}
\tag{3.3.40}
$$

where the Gronwall term $C_{G,t}$ is defined by

$$
\begin{aligned}
C_{G,t} &\sim \exp\left(\frac{T}{1-K}\right) \\
K &:= C\Delta t \Bigg(\frac{h^{4k_u} + \nu^2 (\Delta t)^{4l}}{(\Delta t)^2 \nu^5} + \frac{1}{\nu} \\
&\qquad + \max_{1\le n\le m} \max_{M\in\mathcal{M}_h} \left\{ \frac{(\tau_M^n)^2}{\nu h^{2d}} \right\} + \max_{1\le n\le m} \max_{M\in\mathcal{M}_h} \left\{ \frac{\tau_M^n}{h^d} \right\} \Bigg)
\end{aligned}
$$

provided that $K < 1$. □

3.3.5 Error Estimates for the Time Discretization

According to our strategy we have split the error into linear and nonlinear errors. Hence, we just combine these estimates to get the final result.

Lemma 3.3.9. *For all $1 \le m \le N$ the error due to time discretization can be bounded by*

$$
\begin{aligned}
\|\widetilde{\boldsymbol{\xi}}_u^m\|_0^2 \le C_{G,t} \Bigg(&\frac{(\Delta t)^4 \max_{1\le n\le m} \max_{M\in\mathcal{M}_h} \{\tau_M^n\}^2}{\nu^3 h^{2d}} \\
&+ \frac{(\Delta t)^4 \max_{1\le n\le m} \max_{M\in\mathcal{M}_h} \{\tau_M^n\}}{\nu^2 h^d} \\
&+ (\Delta t)^2 \max_{1\le n\le N} \max_{M\in\mathcal{M}_h} \{\tau_M^n |\widetilde{\boldsymbol{u}}_M^n|^2\} \\
&+ \Delta t \frac{h^{2k_u}}{\nu^3} + \frac{(\Delta t)^{2l+1}}{\nu^2} + \frac{(\Delta t)^4}{\nu^3} \Bigg).
\end{aligned}
\tag{3.3.41}
$$

$$\begin{aligned}
&\Delta t \sum_{n=1}^{m} \Big[\nu\|\nabla\widetilde{\boldsymbol{\xi}}_u^n\|_0^2 + \gamma\|\nabla\cdot\widetilde{\boldsymbol{\xi}}_u^n\|_0^2 + \sum_{M\in\mathcal{M}_h} \tau_M^n\|\kappa_M((\widetilde{\boldsymbol{u}}_M^n\cdot\nabla)\widetilde{\boldsymbol{\xi}}_u^n)\|_{0,M}^2\Big] \\
&\le C_{G,t}\Bigg(\frac{(\Delta t)^4 \max_{1\le n\le m}\max_{M\in\mathcal{M}_h}\{\tau_M^n\}^2}{\nu^3 h^{2d}} \\
&\qquad + \frac{(\Delta t)^4 \max_{1\le n\le m}\max_{M\in\mathcal{M}_h}\{\tau_M^n\}}{\nu^2 h^d} \\
&\qquad + (\Delta t)^2 \max_{1\le n\le N}\max_{M\in\mathcal{M}_h}\{\tau_M^n|\widetilde{\boldsymbol{u}}_M^n|^2\} \\
&\qquad + \Delta t\frac{h^{2k_u}}{\nu^3} + \frac{(\Delta t)^{2l+1}}{\nu^2} + \frac{(\Delta t)^4}{\nu^3}\Bigg) + C(\Delta t)^2.
\end{aligned} \tag{3.3.42}$$

Proof. For the linear error we have according to Theorem 3.3.4 the estimate

$$\Delta t\sum_{i=1}^{m}\|\widetilde{\boldsymbol{\eta}}_u^i\|_0^2 + \frac{(\Delta t)^3}{\nu^2}\sum_{i=1}^{m}\Big(\gamma\|\nabla\cdot\widetilde{\boldsymbol{\eta}}_u^i\|_0^2 + \nu\|\nabla\widetilde{\boldsymbol{\eta}}_u^i\|_0^2\Big) \le C\frac{(\Delta t)^4}{\nu^2} \tag{3.3.43}$$

and for the nonlinear error we obtain

$$\begin{aligned}
&\|\widetilde{\boldsymbol{e}}_u^m\|_0^2 + \Delta t\sum_{n=1}^{m}\Big[\nu\|\nabla\widetilde{\boldsymbol{e}}_u^n\|_0^2 + \gamma\|\nabla\cdot\widetilde{\boldsymbol{e}}_u^n\|_0^2 \\
&\qquad\qquad + \sum_{M\in\mathcal{M}_h}\tau_M^n\|\kappa_M((\widetilde{\boldsymbol{u}}_M^n\cdot\nabla)\widetilde{\boldsymbol{e}}_u^n)\|_{0,M}^2\Big] \\
&\le C_{G,t}\Bigg(\frac{(\Delta t)^4\max_{1\le n\le m}\max_{M\in\mathcal{M}_h}\{\tau_M^n\}^2}{\nu^3 h^{2d}} \\
&\qquad + \frac{(\Delta t)^4\max_{1\le n\le m}\max_{M\in\mathcal{M}_h}\{\tau_M^n\}}{\nu^2 h^d} \\
&\qquad + (\Delta t)^2\max_{1\le n\le N}\max_{M\in\mathcal{M}_h}\{\tau_M^n|\widetilde{\boldsymbol{u}}_M^n|^2\} \\
&\qquad + \frac{\Delta t h^{2k_u}}{\nu^3} + \frac{(\Delta t)^{2l+1}}{\nu^2} + \frac{(\Delta t)^4}{\nu^3}\Bigg).
\end{aligned} \tag{3.3.44}$$

We further notice

$$\begin{aligned}
\tau_M^n\|\kappa_M((\widetilde{\boldsymbol{u}}_{ht}^n\cdot\nabla)\widetilde{\boldsymbol{\xi}}_u^n)\|_0^2 &\le C\tau_M^n\|\kappa_M((\widetilde{\boldsymbol{u}}_{ht}^n\cdot\nabla)\widetilde{\boldsymbol{\eta}}_u^n)\|_0^2 + C\tau_M^n\|\kappa_M((\widetilde{\boldsymbol{u}}_{ht}^n\cdot\nabla)\widetilde{\boldsymbol{e}}_u^n)\|_0^2 \\
&\le C\tau_M^n|\widetilde{\boldsymbol{u}}_{ht}^n|^2\|\widetilde{\boldsymbol{\eta}}_u^n\|_1^2 + C\tau_M^n\|\kappa_M((\widetilde{\boldsymbol{u}}_M^n\cdot\nabla)\widetilde{\boldsymbol{e}}_u^n)\|_0^2
\end{aligned}$$

Hence, we get in combination the proposed result.

□

3.4 Error Estimates for Spatial-Temporal Discretization

For the sake of brevity we define the total error by

$$\widetilde{\boldsymbol{\zeta}}_u^n := \boldsymbol{u}(t_n) - \widetilde{\boldsymbol{u}}_{ht}^n \quad \boldsymbol{\zeta}_u^n := \boldsymbol{u}(t_n) - \boldsymbol{u}_{ht}^n \quad \widetilde{\boldsymbol{\zeta}}_p^n := p(t_n) - p_{ht}^n.$$

Now that we derived estimates for both the discretization in time and space we can combine these results into the following theorem.

Theorem 3.4.1. *If the stabilization parameters satisfy for all* $1 \leq n \leq N$

$$\gamma = \gamma_0, \quad \nu Re_M^2 \leq 1, \quad \tau_M^n \lesssim \min\left\{ \frac{(\Delta t)^2}{|\widetilde{\boldsymbol{u}}_M^n|^2 h^{2(s-k_u)}}, h^d \right\},$$

the approximation orders fulfill $k_u = k_p + 1$ *and the time step size and the mesh width are chosen according to*

$$\begin{aligned} \Delta t &\leq \min\{\nu^{3/2}, h^{d/4}\} \\ h^{4k_u} &\leq \nu(\Delta t) \min\{\nu \Delta t, \nu^4\}, \end{aligned}$$

the error due to spatial and discretization in time can be bounded by

$$\begin{aligned} \|\boldsymbol{\zeta}_u\|^2_{l^\infty(t_0,T;[L^2(\Omega)]^d)} &\leq C(\Delta t)^4 + Ch^{2k_u} \\ \|\boldsymbol{\zeta}_u\|^2_{l^2(t_0,T;LPS)} &\leq C(\Delta t)^2 + Ch^{2k_u} \\ \|\zeta_p\|^2_{l^2(t_0,T;L^2(\Omega))} &\leq \frac{\|\boldsymbol{\zeta}_u\|^2_{l^\infty(t_0,T;[L^2(\Omega)]^d)}}{(\Delta t)^2} + \|\boldsymbol{\zeta}_u\|^2_{l^2(t_0,T;LPS)} \\ &\leq C\frac{(\Delta t)^4 + Ch^{2k_u}}{(\Delta t)^2} + C(\Delta t)^2 + Ch^{2k_u}. \end{aligned}$$

Proof. For the velocity error we use the fact that all the error terms behave linear. Due to Corollary 3.2.3 a discrete version of the spatial results reads

$$\begin{aligned} \|\boldsymbol{\xi}_{u,h}\|^2_{l^\infty(t_0,T;[L^2(\Omega)]^d)} &\leq \|\boldsymbol{\xi}_{u,h}\|^2_{L^\infty(t_0,T;[L^2(\Omega)]^d)} \\ \|\boldsymbol{\xi}_{u,h}\|^2_{l^2(t_0,T;LPS)} &\leq C\|\boldsymbol{\xi}_{u,h}\|^2_{L^2(t_0,T;LPS)} + C(\Delta t)^{2l}. \end{aligned}$$

where the right-hand sides are bounded by

$$\begin{aligned} &\|\boldsymbol{\xi}_{u,h}\|^2_{L^\infty(t_0,T;[L^2(\Omega)]^d)} + \|\boldsymbol{\xi}_{u,h}\|^2_{L^2(t_0,T;LPS)} \\ &\leq C\sum_M h^{2k_u} \int_{t_0}^T e^{C_G(\boldsymbol{u})(t-\tau)} \Big(\Big(1 + \nu Re_M^2 + \tau_M |\boldsymbol{u}_M|^2 + d\gamma\Big) |\boldsymbol{u}(\tau)|^2_{W^{k_u+1,2}(\omega_M)} \\ &\quad + \tau_M |\boldsymbol{u}_M|^2 h^{2(s-k_u)} |\boldsymbol{u}(\tau)|^2_{W^{s+1,2}(\omega_M)} + |\partial_t \boldsymbol{u}(\tau)|^2_{W^{k_u,2}(\omega_M)} \\ &\quad + h^{2(k_p+1-k_u)} \min\left(\frac{d}{\nu}, \frac{1}{\gamma}\right) |p(\tau)|^2_{W^{k_p+1,2}(\omega_M)} \Big) d\tau. \end{aligned}$$

Furthermore we estimate the spatial error for the time-discretized LPS-term by

$$s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \xi_{u,h}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \xi_{u,h}^n) \leq \max_M \{\tau_M^n |\widetilde{\boldsymbol{u}}_M^n|^2\} |||\xi_{u,h}^n|||_1^2.$$

In combination, we obtain

$$\|\boldsymbol{\zeta}_u\|^2_{l^\infty(t_0,T;[L^2(\Omega)]^d)} \leq C\|\boldsymbol{\xi}_u\|^2_{l^\infty(t_0,T;[L^2(\Omega)]^d)} + C\|\boldsymbol{\xi}_{u,h}\|^2_{l^\infty(t_0,T;[L^2(\Omega)]^d)}$$

$$
\begin{aligned}
&\leq C \sum_M h^{2k_u} \int_{t_0}^T e^{C_G(\boldsymbol{u})(t-\tau)} \Big(\Big(1 + \nu Re_M^2 + \tau_M |\boldsymbol{u}_M|^2 + d\gamma\Big) |\boldsymbol{u}(\tau)|^2_{W^{k_u+1,2}(\omega_M)} \\
&\quad + \tau_M |\boldsymbol{u}_M|^2 h^{2(s-k_u)} |\boldsymbol{u}(\tau)|^2_{W^{s+1,2}(\omega_M)} + |\partial_t \boldsymbol{u}(\tau)|^2_{W^{k_u,2}(\omega_M)} \\
&\quad + h^{2(k_p+1-k_u)} \min\left(\frac{d}{\nu}, \frac{1}{\gamma}\right) |p(\tau)|^2_{W^{k_p+1,2}(\omega_M)} \Big) d\tau + C(\Delta t)^{2l} \\
&\quad + C_{G,t} \Bigg(\frac{(\Delta t)^4 \max_{1\leq n\leq N} \max_{M\in\mathcal{M}_h} \{\tau_M^n\}^2}{\nu^3 h^{2d}} \\
&\qquad\quad + \frac{(\Delta t)^4 \max_{1\leq n\leq N} \max_{M\in\mathcal{M}_h} \{\tau_M^n\}}{\nu^2 h^d} \\
&\qquad\quad + \frac{(\Delta t)^2}{\nu^2} \max_{1\leq n\leq N} \max_{M\in\mathcal{M}_h} \{\tau_M^n |\widetilde{\boldsymbol{u}}_M^n|^2\} \\
&\qquad\quad + \Delta t \frac{h^{2k_u}}{\nu^3} + \frac{(\Delta t)^{2l+1}}{\nu^2} + \frac{(\Delta t)^4}{\nu^3} \Bigg) =: \chi_{u1}
\end{aligned}
$$

$$
\begin{aligned}
&\|\boldsymbol{\zeta}_u\|^2_{l^2(t_0,T;LPS)} \leq \|\boldsymbol{\xi}_u\|^2_{l^2(t_0,T;LPS)} + \|\boldsymbol{\xi}_{u,h}\|^2_{l^2(t_0,T;LPS)} \\
&\leq C \sum_M h^{2k_u} \int_{t_0}^T e^{C_G(\boldsymbol{u})(t-\tau)} \Big(\Big(1 + \nu Re_M^2 + \tau_M |\boldsymbol{u}_M|^2 + d\gamma\Big) |\boldsymbol{u}(\tau)|^2_{W^{k_u+1,2}(\omega_M)} \\
&\quad + \tau_M |\boldsymbol{u}_M|^2 h^{2(s-k_u)} |\boldsymbol{u}(\tau)|^2_{W^{s+1,2}(\omega_M)} + |\partial_t \boldsymbol{u}(\tau)|^2_{W^{k_u,2}(\omega_M)} \\
&\quad + h^{2(k_p+1-k_u)} \min\left(\frac{d}{\nu}, \frac{1}{\gamma}\right) |p(\tau)|^2_{W^{k_p+1,2}(\omega_M)} \Big) d\tau + C(\Delta t)^{2l} + C(\Delta t)^2 \\
&\quad + C_{G,t} \Bigg(\frac{(\Delta t)^4 \max_{1\leq n\leq N} \max_{M\in\mathcal{M}_h} \{\tau_M^n\}^2}{\nu^3 h^{2d}} \\
&\qquad\quad + \frac{(\Delta t)^4 \max_{1\leq n\leq N} \max_{M\in\mathcal{M}_h} \{\tau_M^n\}}{\nu^2 h^d} \\
&\qquad\quad + \frac{(\Delta t)^2}{\nu^2} \max_{1\leq n\leq N} \max_{M\in\mathcal{M}_h} \{\tau_M^n |\widetilde{\boldsymbol{u}}_M^n|^2\} \\
&\qquad\quad + \Delta t \frac{h^{2k_u}}{\nu^3} + \frac{(\Delta t)^{2l+1}}{\nu^2} + \frac{(\Delta t)^4}{\nu^3} \Bigg) =: \chi_{u2}
\end{aligned}
$$

In order to balance the stabilization parameters we bound the streamline velocity approximation in the spatial discretization by the following argument. Choose the partitioning $\mathcal{M}_h := ([t_0 + (n-1)\Delta t, t_0 + n\Delta t])_{n=1,\ldots,(T-t_0)/\Delta t}$ and a piecewise linear nodal basis ϕ_n. Define $\hat{\boldsymbol{u}}_{ht}(t)$ by the finite element approximation $\hat{\boldsymbol{u}}_{ht} := \sum_n \widetilde{\boldsymbol{u}}_{ht}^n \phi_n$ and choose τ_M as the finite element approximation of τ_M^n according to $\tau_M := \sum_n \tau_M^n \phi_n$. In particular, $\hat{\boldsymbol{u}}_{ht}, \tau_M \in H^l(t_0, T; L^2(\Omega))$ and $\tau_M(t_n) = \tau_M^n$. With $\hat{\boldsymbol{u}}_M(t) := \frac{1}{|M|}\int_M \hat{\boldsymbol{u}}_{ht}(t)dx$ and the abbreviation $K := \|\boldsymbol{u}\|^2_{L^\infty(t_0,T;[W^{k_u+1,2}(\omega_M)]^d)}$ it holds

$$
\int_{t_0}^T \tau_M(t) |\boldsymbol{u}(t)|^2_{W^{k_u+1,2}(\omega_M)} |\boldsymbol{u}_M(t)|^2 dt
$$

$$
\begin{aligned}
&\le C\int_{t_0}^{T}\tau_M(t)|\boldsymbol{u}(t)|^2_{W^{k_u+1,2}(\omega_M)}|\hat{\boldsymbol{u}}_M(t)|^2dt\\
&\quad + C\int_{t_0}^{T}\tau_M(t)|\boldsymbol{u}(t)|^2_{W^{k_u+1,2}(\omega_M)}|\boldsymbol{u}_M(t)-\hat{\boldsymbol{u}}_M(t)|^2dt\\
&\le CK\|\sqrt{\tau_M}\hat{\boldsymbol{u}}_M\|^2_{L^2(t_0,T)}\\
&\quad + Ch^{-d}\int_{t_0}^{T}\tau_M(t)|\boldsymbol{u}(t)|^2_{W^{k_u+1,2}(\omega_M)}\|\boldsymbol{u}(t)-\hat{\boldsymbol{u}}_{ht}(t)\|^2_{0,M}dt\\
&\le CK\|\sqrt{\tau_M}\hat{\boldsymbol{u}}_M\|^2_{L^2(t_0,T)} + Ch^{-d}K\|\sqrt{\tau_M}(\boldsymbol{u}_h-\hat{\boldsymbol{u}}_{ht})\|^2_{L^2(t_0,T;[L^2(M)]^d)}\\
&\le CK\|\sqrt{\tau_M}\widetilde{\boldsymbol{u}}_M\|^2_{l^2(t_0,T)}\\
&\quad + Ch^{-d}K\Big(\|\sqrt{\tau_M}(\boldsymbol{u}_h-\widetilde{\boldsymbol{u}}_{ht})\|^2_{l^2(t_0,T;[L^2(M)]^d)} + C(\Delta t)^{2l}\Big)\\
&\le CK\Big(\max_{1\le n\le N}\{\tau_M^n|\widetilde{\boldsymbol{u}}_M|^2\} + \frac{\max_{1\le n\le N}\{\tau_M^n\}}{h^d}\|\widetilde{\boldsymbol{\xi}}_u\|^2_{l^2(t_0,T;[L^2(M)]^d)} + C(\Delta t)^{2l}\Big)\\
&\le C\|\boldsymbol{u}\|^2_{L^\infty(t_0,T;[W^{k_u+1,2}(\omega_M)]^d)}\\
&\quad\Big(\max_{1\le n\le N}\{\tau_M^n|\widetilde{\boldsymbol{u}}_M|^2\} + \frac{\max_{1\le n\le N}\{\tau_M^n\}}{h^d}(\Delta t)^4 + C(\Delta t)^{2l}\Big).
\end{aligned}
$$

Using the parameter choice

$$
\gamma=\gamma_0,\quad \nu Re_M^2\le 1,\quad \tau_M^n\lesssim\min\left\{\frac{(\Delta t)^2}{|\widetilde{\boldsymbol{u}}_M^n|^2h^{2(s-k_u)}},h^d\right\}
$$

we obtain the claim for $\|\widetilde{\boldsymbol{\zeta}}_u\|_{l^2(t_0,T;[L^2(\Omega)]^d)}$ and $\|\widetilde{\boldsymbol{\zeta}}_u\|_{l^2(t_0,T;LPS)}$, if $l=2$ and we require $\boldsymbol{u}_h\in W^{2,2}(t_0,T;[L^2(\Omega)]^d)$.

In order to obtain the estimate for the pressure error in the $L^2(\Omega)$-norm we utilize the discrete inf-sup stability of the ansatz spaces, i.e.

$$
\exists\boldsymbol{w}_h\in\boldsymbol{V}_h\colon\ \|\nabla\boldsymbol{w}_h\|_0\le\|\zeta_p^n\|_0/\beta,\quad -(\nabla\cdot\boldsymbol{w}_h,\zeta_p^n)=\|\zeta_p^n\|_0^2 \tag{3.4.1}
$$

We test the convection-diffusion error equation with $\boldsymbol{w}_h$:

$$
\begin{aligned}
&\left(\frac{3\widetilde{\boldsymbol{\zeta}}_u^n-4\boldsymbol{\zeta}_u^{n-1}+\boldsymbol{\zeta}_u^{n-2}}{2\Delta t},\boldsymbol{w}_h\right)+\nu(\nabla\widetilde{\boldsymbol{\zeta}}_u^n,\nabla\boldsymbol{w}_h)+\gamma(\nabla\cdot\widetilde{\boldsymbol{\zeta}}_u^n,\nabla\cdot\boldsymbol{w}_h)\\
&=-c(\boldsymbol{u}(t_n),\boldsymbol{u}(t_n),\boldsymbol{w}_h)+c(\widetilde{\boldsymbol{u}}_{ht}^n,\widetilde{\boldsymbol{u}}_{ht}^n,\boldsymbol{w}_h)+s_h(\widetilde{\boldsymbol{u}}_{ht}^n,\widetilde{\boldsymbol{u}}_{ht}^n,\widetilde{\boldsymbol{u}}_{ht}^n,\boldsymbol{w}_h)\\
&\quad+(D_t\boldsymbol{u}(t_n)-\partial_t\boldsymbol{u}(t_n),\boldsymbol{w}_h)-(\nabla(p(t_n)-p_{ht}^{n-1}),\boldsymbol{w}_h).
\end{aligned} \tag{3.4.2}
$$

where $D_t\boldsymbol{u}(t_n):=(3\boldsymbol{u}(t_n)-4\boldsymbol{u}(t_{n-1})+\boldsymbol{u}(t_{n-2}))/(2\Delta t)$ and $\partial_t\boldsymbol{u}$ is the time derivative of $\boldsymbol{u}$.

Noticing $\|\boldsymbol{f}\|_{-1}\le\|\boldsymbol{f}\|_0$ we obtain

$$
\begin{aligned}
&\|\nabla\boldsymbol{w}_h\|_0\|\zeta_p^{n-1}\|_0\le\frac{1}{\beta}\|\zeta_p^{n-1}\|_0^2=-(\nabla\zeta_p^{n-1},\boldsymbol{w}_h)\\
&\le\left\|\frac{3\widetilde{\boldsymbol{\zeta}}_u^n-4\boldsymbol{\zeta}_u^{n-1}+\boldsymbol{\zeta}_u^{n-2}}{2\Delta t}\right\|_{-1}\|\nabla\boldsymbol{w}_h\|_0+\nu\|\nabla\widetilde{\boldsymbol{\zeta}}_u^n\|_0\|\nabla\boldsymbol{w}_h\|_0
\end{aligned}
$$

$$
\begin{aligned}
&+ \gamma \|\nabla \cdot \widetilde{\boldsymbol{\zeta}}_u^n\|_0 \|\nabla \cdot \boldsymbol{w}_h\|_0 \\
&+ c(\boldsymbol{u}(t_n), \boldsymbol{u}(t_n), \boldsymbol{w}_h) - c(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{w}_h) + s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{w}_h) \\
&+ \|D_t \boldsymbol{u}(t_n) - \partial_t \boldsymbol{u}(t_n)\|_{-1} \|\nabla \boldsymbol{w}_h\|_0 + \|p(t_n) - p(t_{n-1})\|_0 \|\nabla \cdot \boldsymbol{w}_h\|_0.
\end{aligned}
$$

We calculate for the convective terms

$$
\begin{aligned}
&c(\boldsymbol{u}(t_n), \boldsymbol{u}(t_n), \boldsymbol{w}_h) - c(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{w}_h) = c(\widetilde{\boldsymbol{\zeta}}_u^n, \boldsymbol{u}(t_n), \boldsymbol{w}_h) - c(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{\zeta}}_u^n, \boldsymbol{w}_h) \\
&= c(\widetilde{\boldsymbol{\zeta}}_u^n, \boldsymbol{u}(t_n), \boldsymbol{w}_h) - c(\boldsymbol{u}(t_n), \widetilde{\boldsymbol{\zeta}}_u^n, \boldsymbol{w}_h) - c(\widetilde{\boldsymbol{\zeta}}_u^n, \widetilde{\boldsymbol{\zeta}}_u^n, \boldsymbol{w}_h) \\
&\le C \|\widetilde{\boldsymbol{\zeta}}_u^n\|_0 \|\boldsymbol{u}(t_n)\|_2 \|\boldsymbol{w}_h\|_1 + C \|\widetilde{\boldsymbol{\zeta}}_u^n\|_1^2 \|\boldsymbol{w}_h\|_1
\end{aligned}
$$

and for the nonlinear stabilization

$$
\begin{aligned}
&s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{w}_h) = s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{u}(t_n) - \widetilde{\boldsymbol{\zeta}}_u^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{w}_h) \\
&\le C \sum_{M \in \mathcal{M}_h} \tau_M^n |\widetilde{\boldsymbol{u}}_M^n|^2 \|\kappa_M(\boldsymbol{u}(t_n))\|_{0,M} \|\|\boldsymbol{w}_h\|_{1,M} \\
&\quad + C \sum_{M \in \mathcal{M}_h} \tau_M^n \|\kappa_M((\widetilde{\boldsymbol{u}}_M^n \cdot \nabla) \widetilde{\boldsymbol{\zeta}}_u^n)\|_{0,M} |\widetilde{\boldsymbol{u}}_M^n| \|\boldsymbol{w}_h\|_{1,M} \\
&\le C \Bigg(\max_{M \in \mathcal{M}_h} \{\tau_M^n |\widetilde{\boldsymbol{u}}_M^n|^2 \|\kappa_M(\boldsymbol{u}(t_n))\|_{0,M}\} \\
&\qquad + \sum_{M \in \mathcal{M}_h} \tau_M^n |\widetilde{\boldsymbol{u}}_M^n| \|\kappa_M((\widetilde{\boldsymbol{u}}_M^n \cdot \nabla) \widetilde{\boldsymbol{\zeta}}_u^n)\|_{0,M} \Bigg) \|\nabla \boldsymbol{w}_h\|_0.
\end{aligned}
$$

We combine these results and obtain due to the approximation property of κ_M and the estimates for $\|\widetilde{\boldsymbol{\zeta}}_u\|_{l^\infty(t_0,T;[L^2(\Omega)]^d)}$ and $\|\widetilde{\boldsymbol{\zeta}}_u\|_{l^2(t_0,T;LPS)}$:

$$
\begin{aligned}
&\Delta t \sum_{n=1}^N \|\zeta_p^{n-1}\|_0^2 \le C \Bigg\{ \frac{1}{(\Delta t)^2} \|\widetilde{\boldsymbol{\zeta}}_u\|_{l^\infty(t_0,T;[L^2(\Omega)]^d)}^2 + \nu^2 \|\nabla \widetilde{\boldsymbol{\zeta}}_u\|_{l^2(t_0,T;[L^2(\Omega)]^d)}^2 \\
&\quad + \gamma^2 \|\nabla \cdot \widetilde{\boldsymbol{\zeta}}_u\|_{l^2(t_0,T;[L^2(\Omega)]^d)}^2 + \|\widetilde{\boldsymbol{\zeta}}_u^n\|_{l^\infty(t_0,T;[L^2(\Omega)]^d)}^2 \|\boldsymbol{u}\|_{l^2(t_0,T;[H^2(\Omega)]^d)}^2 \\
&\quad + \|\widetilde{\boldsymbol{\zeta}}_u^n\|_{l^\infty(t_0,T;[H^1(\Omega)]^d)}^2 \|\widetilde{\boldsymbol{\zeta}}_u^n\|_{l^2(t_0,T;[H^1(\Omega)]^d)}^2 \\
&\quad + \max_{1 \le n \le N} \max_{M \in \mathcal{M}_h} \{\tau_M^n |\widetilde{\boldsymbol{u}}_M^n|^2\}^2 h^{2k_u} \|\boldsymbol{u}\|_{l^2(t_0,T;[W^{k_u+1,2}(\Omega)]^d)}^2 \\
&\quad + \max_{1 \le n \le N} \max_{M \in \mathcal{M}_h} \{\tau_M^n\} \Delta t \sum_{n=1}^N \sum_{M \in \mathcal{M}_h} \tau_M^n |\widetilde{\boldsymbol{u}}_M^n|^2 \|\kappa_M((\widetilde{\boldsymbol{u}}_M^n \cdot \nabla) \widetilde{\boldsymbol{\zeta}}_u^n)\|_{0,M}^2 + (\Delta t)^2 \Bigg\} \\
&\le C \left(\frac{1}{(\Delta t)^2} + \|\boldsymbol{u}\|_{l^2(t_0,T;[H^2(\Omega)]^d)}^2 \right) \chi_{u1} \\
&\quad + \left(\nu + \gamma + \max_{1 \le n \le N} \max_{M \in \mathcal{M}_h} \{\tau_M^n |\widetilde{\boldsymbol{u}}_M^n|^2\} \right) \chi_{u2} \\
&\quad + C \max_{1 \le n \le N} \max_{M \in \mathcal{M}_h} \{\tau_M^n |\widetilde{\boldsymbol{u}}_M^n|^2\}^2 h^{2k_u} \|\boldsymbol{u}\|_{l^2(t_0,T;[W^{k_u+1,2}(\Omega)]^d)}^2 + C(\Delta t)^2 \\
&\quad + C \frac{\chi_{u2}^2}{\nu^2 \Delta t}
\end{aligned}
$$

Using the parameter choice

$$\gamma = \gamma_0, \quad \nu Re_M^2 \leq 1, \quad \tau_M^n \lesssim \min\left\{ \frac{(\Delta t)^2}{|\widetilde{\boldsymbol{u}}_M^n|^2 h^{2(s-k_u)}}, h^d \right\}$$

and choosing the time step size according to $\Delta t \leq \min\{\nu^{3/2}, h^{d/4}\}$, and the mesh width as $h^{4k_u} \leq \nu(\Delta t)\min\{\nu\Delta t, \nu^4\}$ the error estimates can be written as

$$\begin{aligned}
\|\boldsymbol{\zeta}_u\|^2_{l^\infty(t_0,T;[L^2(\Omega)]^d)} &\leq C(\Delta t)^4 + Ch^{2k_u} \\
\|\boldsymbol{\zeta}_u\|^2_{l^2(t_0,T;LPS)} &\leq C(\Delta t)^2 + Ch^{2k_u} \\
\|\zeta_p\|^2_{l^2(t_0,T;L^2(\Omega))} &\leq \frac{\|\boldsymbol{\zeta}_u\|^2_{l^\infty(t_0,T;[L^2(\Omega)]^d)}}{(\Delta t)^2} + \|\boldsymbol{\zeta}_u\|^2_{l^2(t_0,T;LPS)} \\
&\leq C\frac{(\Delta t)^4 + Ch^{2k_u}}{(\Delta t)^2} + C(\Delta t)^2 + Ch^{2k_u}.
\end{aligned}$$

□

Remark 3.4.2. If we want to bound the pressure error estimate and the $L^2(\Omega)$-norm of the velocity best, we should choose $(\Delta t)^2 \sim h^{k_u} \leq h^{d/2}$. On the other hand, if we are most interested in the LPS-norm of the velocity, we should use $(\Delta t)^2 \sim h^{2k_u}$. In fact, this is the only error that has the expected orders of convergence. If we restrict ourselves to this case, the parameter choice changes a bit and we get

$$\begin{aligned}
\gamma = \gamma_0, \quad \nu Re_M^2 \leq 1, \quad \Delta t \leq \min\{\nu^3, h^{d/4}\} \\
\max_{1\leq n\leq N} \tau_M^n |\boldsymbol{u}_M^n|^2 \leq \min\{1, h^d/(\Delta t)^2\}.
\end{aligned}$$

In either case we observe that the rate of convergence for the errors with respect to time discretization is in agreement with the estimates in the available literature for this scheme. The spatial convergence rates however suffers from a suboptimal estimate for the $L^2(\Omega)$ norm of the velocity. The interpolation operator would yield an approximation error proportional to h^{k_u+1}. If we would obtain such an estimate we could also bound the pressure with the rate of convergence of the interpolation operator provided $k_p = k_u - 1$.

Remark 3.4.3. A main result for the spatial error estimates that we used in this approach is that the Gronwall constant does not dependent explicitly on the viscosity ν. Unfortunately, this result does not transfer to the discretization in time. Due to the fact that we only assumed the time discretized quantities to fulfill the regularity requirements

$$\begin{aligned}
\boldsymbol{u}_h &\in W^{1,\infty}(t_0,T;[L^2(\Omega)]^d) \cap W^{l,2}(t_0,T;[H^1(\Omega)]^d) \\
p_h &\in W^{2,\infty}(t_0,T;H^1(\Omega))
\end{aligned} \tag{3.4.3}$$

we couldn't use an $L^\infty(t_0,T;[H^2(\Omega)]^d)$ for the estimates of the convective term. This results in a Gronwall term according to

$$C_{G,t} \sim \exp\left(\frac{T}{1-K}\right)$$

$$K := C\Delta t\left(\frac{h^{4k_u}+\nu^2(\Delta t)^{4l}}{(\Delta t)^2\nu^5}+\frac{1}{\nu}\right.$$
$$\left.+\max_{1\le n\le m}\max_{M\in\mathcal{M}_h}\left\{\frac{(\tau_M^n)^2}{\nu h^{2d}}\right\}+\max_{1\le n\le m}\max_{M\in\mathcal{M}_h}\left\{\frac{\tau_M^n}{h^d}\right\}\right)$$

Taking the stability requirement $h^{2k_u} \le \nu\Delta t$ into account, this in fact means that we have to choose $\Delta t \lesssim \nu^3$.

If we would have a bound for $\boldsymbol{u}_h$ in $L^\infty(t_0, T; [H^2(\Omega)]^d)$ norm we could improve the behavior in the Gronwall term $C_{G,t}$ with respect to ν:

$$\begin{aligned}
&c(\boldsymbol{u}_h(t_n), \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{e}}_u^n) - c(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{e}}_u^n)\\
&= c(\widetilde{\boldsymbol{\eta}}_u^n + \widetilde{\boldsymbol{e}}_u^n, \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{e}}_u^n) + c(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{\eta}}_u^n + \widetilde{\boldsymbol{e}}_u^n, \widetilde{\boldsymbol{e}}_u^n)\\
&= c(\widetilde{\boldsymbol{\eta}}_u^n + \widetilde{\boldsymbol{e}}_u^n, \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{e}}_u^n) + c(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{e}}_u^n)\\
&= c(\widetilde{\boldsymbol{e}}_u^n, \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{e}}_u^n) + c(\widetilde{\boldsymbol{\eta}}_u^n, \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{e}}_u^n) + c(\boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{e}}_u^n)\\
&\quad - c(\widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{e}}_u^n) - c(\widetilde{\boldsymbol{e}}_u^n, \widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{e}}_u^n)
\end{aligned}\tag{3.4.4}$$

$$\begin{aligned}
&c(\widetilde{\boldsymbol{e}}_u^n, \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{e}}_u^n) \le C\|\widetilde{\boldsymbol{e}}_u^n\|_0\|\boldsymbol{u}_h(t_n)\|_2\|\widetilde{\boldsymbol{e}}_u^n\|_1\\
&\quad\le \frac{\nu}{32}\|\nabla\widetilde{\boldsymbol{e}}_u^n\|_0^2 + \frac{C}{\nu}\|\widetilde{\boldsymbol{e}}_u^n\|_0^2\\
&c(\boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{e}}_u^n) + c(\widetilde{\boldsymbol{\eta}}_u^n, \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{e}}_u^n)\\
&\quad\le C\|\widetilde{\boldsymbol{u}}(t_n)\|_2\|\widetilde{\boldsymbol{\eta}}_u^n\|_0\|\widetilde{\boldsymbol{e}}_u^n\|_1 \le \frac{\nu}{32}\|\nabla\widetilde{\boldsymbol{e}}_u^n\|_0^2 + \frac{C}{\nu}\|\widetilde{\boldsymbol{\eta}}_u^n\|_0^2\\
&c(\widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{e}}_u^n) \le C\|\widetilde{\boldsymbol{\eta}}_u^n\|_1^2\|\widetilde{\boldsymbol{e}}_u^n\|_1 \le \frac{\nu}{32}\|\nabla\widetilde{\boldsymbol{e}}_u^n\|_0^2 + \frac{C}{\nu}\|\widetilde{\boldsymbol{\eta}}_u^n\|_1^4\\
&c(\widetilde{\boldsymbol{e}}_u^n, \widetilde{\boldsymbol{\eta}}_u^n, \widetilde{\boldsymbol{e}}_u^n) \le C\|\widetilde{\boldsymbol{\eta}}_u^n\|_1\|\widetilde{\boldsymbol{e}}_u^n\|_1^2
\end{aligned}\tag{3.4.5}$$

and therefore

$$\begin{aligned}
&c(\boldsymbol{u}_h(t_n), \boldsymbol{u}_h(t_n), \widetilde{\boldsymbol{e}}_u^n) - c(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{e}}_u^n)\\
&\quad\le C\|\nabla\widetilde{\boldsymbol{e}}_u^n\|_0^2\,(\nu + \|\widetilde{\boldsymbol{\eta}}_u^n\|_1) + \frac{C}{\nu}\|\widetilde{\boldsymbol{e}}_u^n\|_0^2 + \frac{C}{\nu}\|\widetilde{\boldsymbol{\eta}}_u^n\|_0^2 + \frac{C}{\nu}\|\widetilde{\boldsymbol{\eta}}_u^n\|_1^4.
\end{aligned}$$

Provided $\|\widetilde{\boldsymbol{\eta}}_u^n\|_1 \le C\nu$ this gives a Gronwall term according to $C_{G,t} \sim \exp\left(\frac{T}{1-\Delta t/\nu}\right)$.

Remark 3.4.4. The restriction on the stabilization parameters

$$\gamma = \gamma_0, \quad \nu Re_M^2 \le 1, \quad \tau_M^n \lesssim \min\left\{\frac{(\Delta t)^2}{|\widetilde{\boldsymbol{u}}_M^n|^2 h^{2(s-k_u)}}, h^d\right\}$$

is the usual one with respect to grad-div stabilization. For the LPS stabilization however we require an extremely small parameter. A choice satisfying this condition would not be expected to show any significant influence on the solution.

Chapter 4

Numerical Results

We comprehend our considerations with some numerical results that show what indeed can be expected for the suggested model.

The considered example is one for which we compute the forcing term $\boldsymbol{f}$ such that

$$\begin{aligned} \boldsymbol{u}(x,y,t) &:= (\sin(1-x)\sin(y+t), \cos(1-x)\cos(y+t))^T \\ p(x,y,t) &:= -\cos(1-x)\sin(y+t) \end{aligned}$$

is the solution to the time-dependent Navier-Stokes problem in the domain $\Omega = [-1,1]^2$ and for $t \in [0,1]$. In Figure 4.1 this reference solution is depicted for the initial and final point in time.

The standard incremental pressure-correction scheme considered in this report is compared with the rotational pressure-correction scheme proposed by Timmermans in [6] and analyzed for the Stokes by Guermon and Shen in [7]. For our setting it reads:

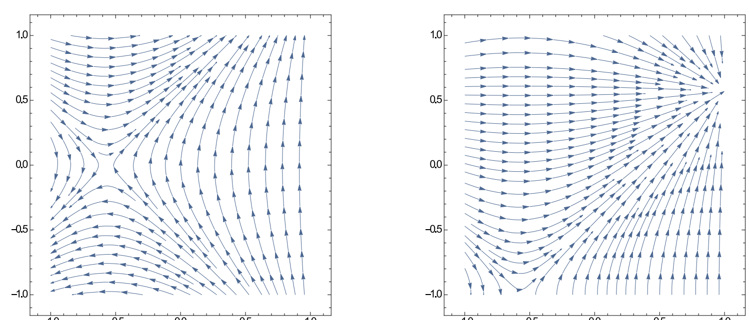

Figure 4.1: Streamlines for the reference solution at $t = 0$ (left) and $t = 1$ (right)

Find $\widetilde{\boldsymbol{u}}_{ht}^n \in \boldsymbol{V}_h, \boldsymbol{u}_{ht}^n \in \boldsymbol{Y}_h$ and $p_{ht}^n \in Q_h$ such that

$$\begin{aligned}
&\left(\frac{3\widetilde{\boldsymbol{u}}_{ht}^n - 4\boldsymbol{u}_{ht}^{n-1} + \boldsymbol{u}_{ht}^{n-2}}{2\Delta t}, \boldsymbol{v}_h\right) + \nu(\nabla \widetilde{\boldsymbol{u}}_{ht}^n, \nabla \boldsymbol{v}_h) + c(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) \\
&\quad + \gamma(\nabla \cdot \widetilde{\boldsymbol{u}}_{ht}^n, \nabla \cdot \boldsymbol{v}_h) + s_h(\widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) \\
&\quad = (\boldsymbol{f}^n, \boldsymbol{v}_h) + (p_{ht}^{n-1}, \nabla \cdot \boldsymbol{v}_h) \\
&\widetilde{\boldsymbol{u}}_{ht}^n|_{\partial\Omega} = 0
\end{aligned} \tag{4.0.1}$$

$$\begin{aligned}
\left(\frac{3\boldsymbol{u}_{ht}^n - 3\widetilde{\boldsymbol{u}}_{ht}^n}{2\Delta t} + \nabla(p_{ht}^n - p_{ht}^{n-1} + \nu\pi_{Q_h}(\nabla \cdot \widetilde{\boldsymbol{u}}_{ht}^n)), \boldsymbol{y}_h\right) &= 0 \\
(\nabla \cdot \boldsymbol{u}_{ht}^n, q_h) &= 0 \\
\boldsymbol{u}_{ht}^n|_{\partial\Omega} &= 0
\end{aligned} \tag{4.0.2}$$

holds for all $\boldsymbol{v}_h \in \boldsymbol{V}_h,\ \boldsymbol{y}_h \in \boldsymbol{Y}_h$ and $q_h \in Q_h$.

In this algorithm π_{Q_h} denotes the $L^2(\Omega)$ projection in the discrete pressure ansatz space Q_h. The modification in the projection equation should prevent that the otherwise artificial boundary condition $\boldsymbol{n} \cdot \nabla p_{ht}^n = \cdots = \boldsymbol{n} \cdot \nabla p_{ht}^0$ deteriorates the error. In fact, it can be shown that for the Stokes problem the error with respect to the $H^1(\Omega)$-norm in the velocity and the $L^2(\Omega)$ norm for the pressure a rate of convergence as $(\Delta t)^{3/2}$ can be expected.
Although we didn't carry out the analysis for this algorithm adapted to our approach, we still believe that similar results hold true and can be observed numerically.

To study the dependence of the error on the diffusion parameter we consider three different Reynolds numbers $Re \in \{10^{-2}, 1, 10^2\}$. Additionally, we want to investigate whether stabilization really improves the results numerically. As a first result we figured out that LPS-SU stabilization does not show any significant influence on the error in the considered parameter regime. Therefore we just consider grad-div stabilization in the following.

For the first results (cf. Figures 4.2, 4.3, 4.4, 4.5) we choose $Re = 10^{-2}$. We see that the error in the considered regime is strictly dominated by the spatial error; with respect to the time step size Δt we see only small influence. The convergence rates with respect to h are as expected of third respectively second order . The effect of the rotational correction is quite big while grad-div stabilization is not required here; for all considered norms the error does not depend on whether we use it or not.

In Figures 4.6, 4.7, 4.8 and 4.9 the case $Re = 1$ is considered. For the errors with respect to the $L^2(\Omega)$ and $H^1(\Omega)$ norm of the velocity we see almost no influence whether we choose the rotational or incremental form or stabilization or not. The convergence rates with respect to spatial discretization is again as expected. This time we observe for both quantities approximately second of order of convergence with respect to time. For the divergence of the velocity we again see a quite big influence of the rotational compared to the standard

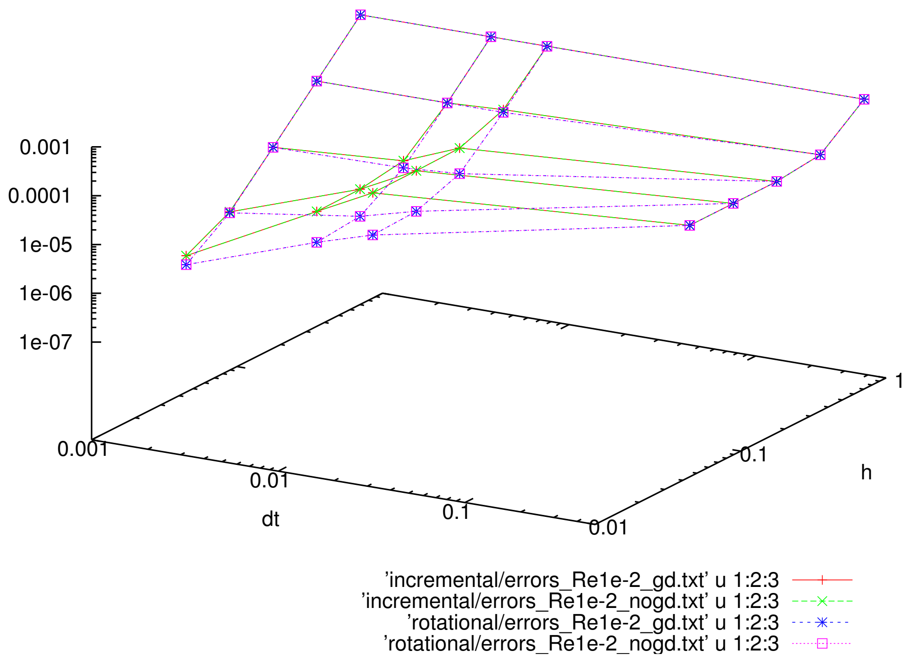

Figure 4.2: $Re = 10^{-2}$, errors for the velocity w.r.t $L^2(\boldsymbol{u})$

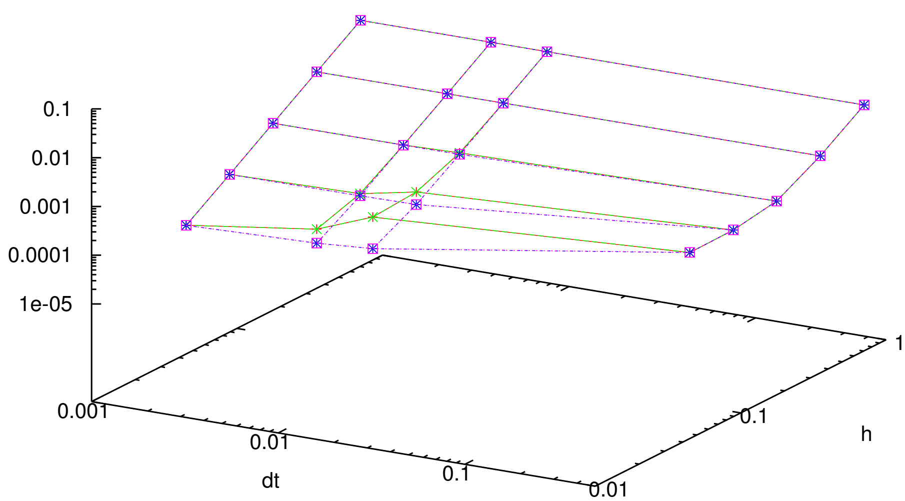

Figure 4.3: $Re = 10^{-2}$, errors for the velocity w.r.t $H^1(\boldsymbol{u})$

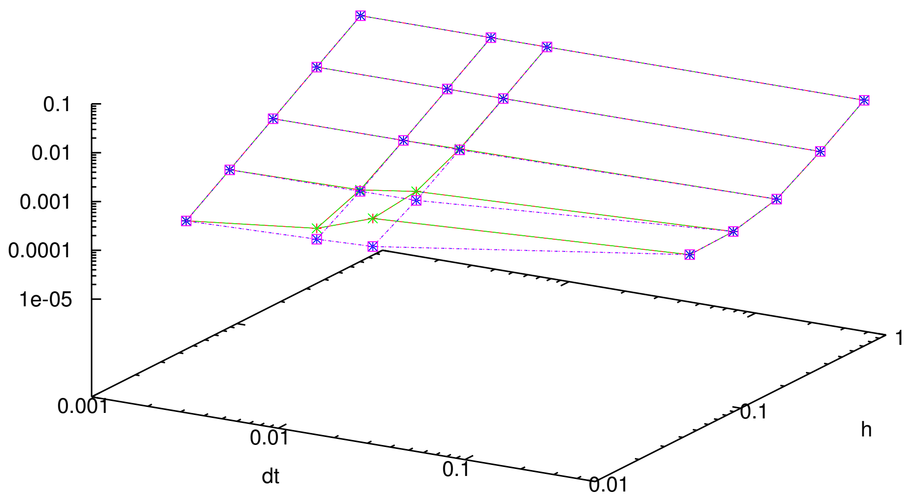

Figure 4.4: $Re = 10^{-2}$, errors for the velocity w.r.t $L^2(\nabla \cdot \boldsymbol{u})$

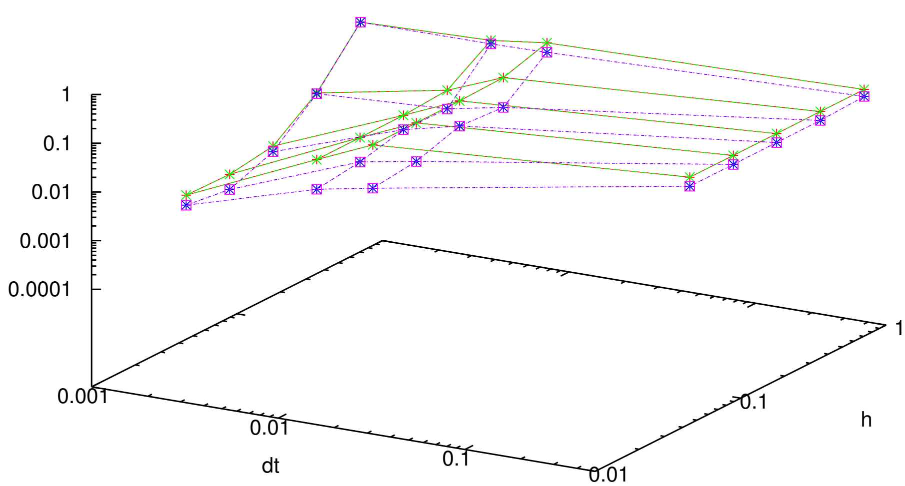

Figure 4.5: $Re = 10^{-2}$, errors for the pressure w.r.t $L^2(p)$

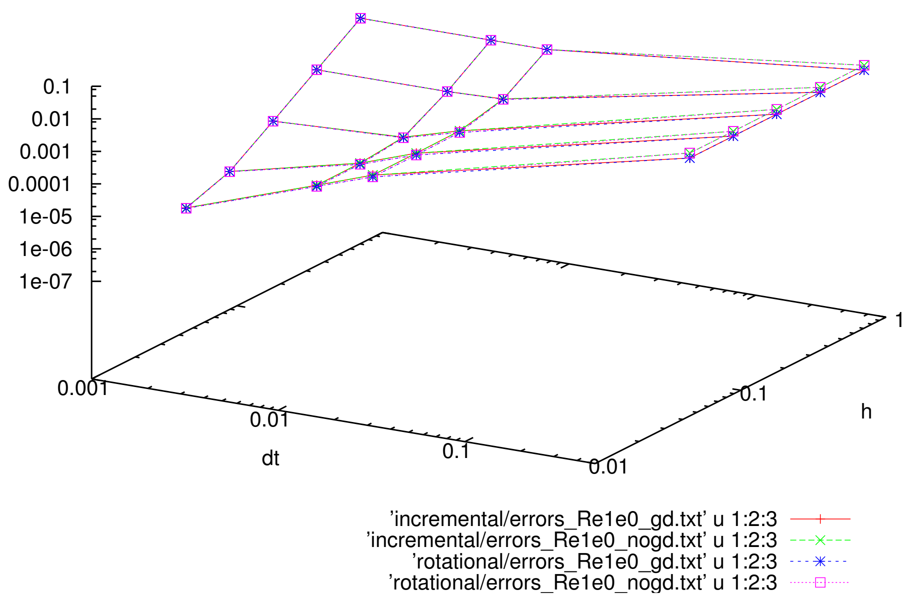

Figure 4.6: $Re = 1$, errors for the velocity w.r.t $L^2(\boldsymbol{u})$

incremental form. Interestingly, the grad-div stabilization is again of minor importance. Finally we for the $L^2(\Omega)$ norm we see four distinct results. Best is the rotational form without stabilization, then the rotational form with stabilization, the incremental form without stabilization and worst the incremental form with stabilization. This in fact the first result in which we see that grad-div stabilization is harmful for an error.

Finally we consider $Re = 10^2$ in Figures 4.10, 4.11, 4.12 and 4.13. For the the first three errors we get a clear picture. Grad-div stabilization diminishes the error by a fixed factor. In fact our analysis tells us in this parameter regime that grad-div stabilization improves the dependence on the Reynolds number Re from Re^1 to $Re^{1/2}$. This is exactly the behavior that we observe. On the other hand the whether we choose the rotational or incremental form is of minor influence. In fact, the correction terms vanishes with decreasing ν. Hence, this is not too surprising. The rates of convergence that we observe are again optimal in the sense that we achieve the rates of the interpolation operators with respect to spatial discretization and for the time discretization a behavior like $(\Delta t)^2$. Note that in view of the analysis carried out for these types of schemes the results are superconvergent with respect to time discretization. For the pressure error the behavior with respect to stabilization is the other way around, similar to the results for $Re = 1$. Again, the error is best when no stabilization is used and the rotational correction is negligible. This last case tells that one really has to know which error one wants to control when

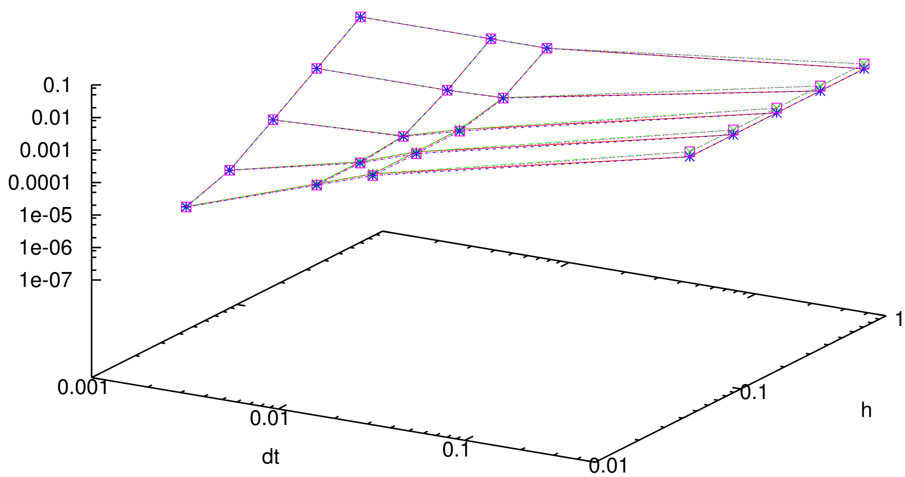

Figure 4.7: $Re = 1$, errors for the velocity w.r.t $H^1(\boldsymbol{u})$

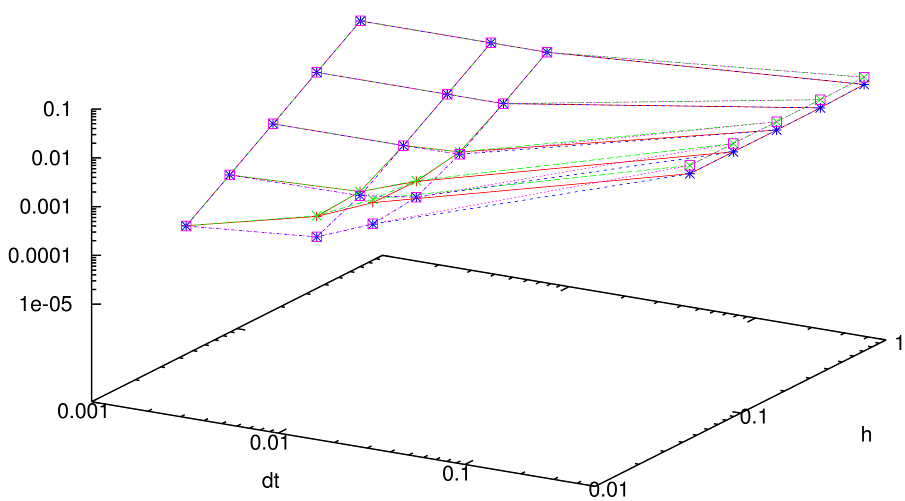

Figure 4.8: $Re = 1$, errors for the velocity w.r.t $L^2(\nabla \cdot \boldsymbol{u})$

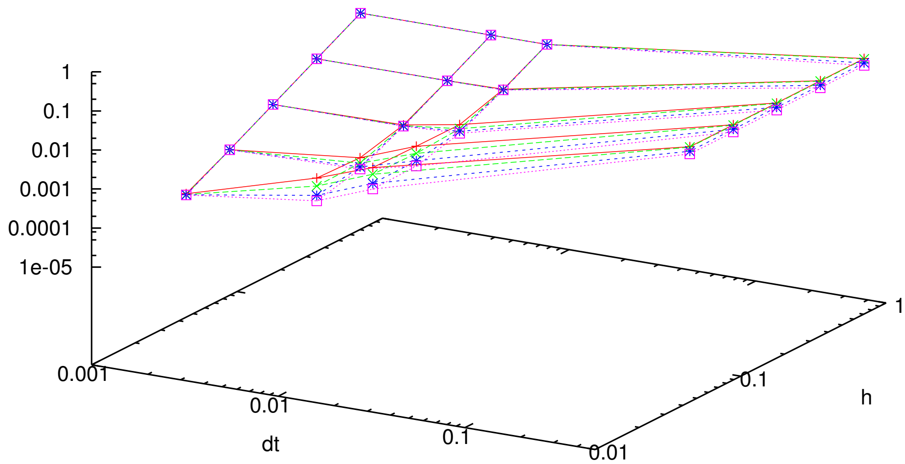

Figure 4.9: $Re = 1$, errors for the pressure w.r.t $H^1(\boldsymbol{u})$

choosing stabilization parameters. There is apparently no choice rule that is best for both velocity and pressure.

In summary one might say, that the rotational correction does never harm and even improves the results considerably if the viscosity ν is not too small. Grad-div stabilization however seems to be beneficial whenever the main interest is in the velocity solution. For the pressure the above example suggests that disabling the stabilization is the best option.

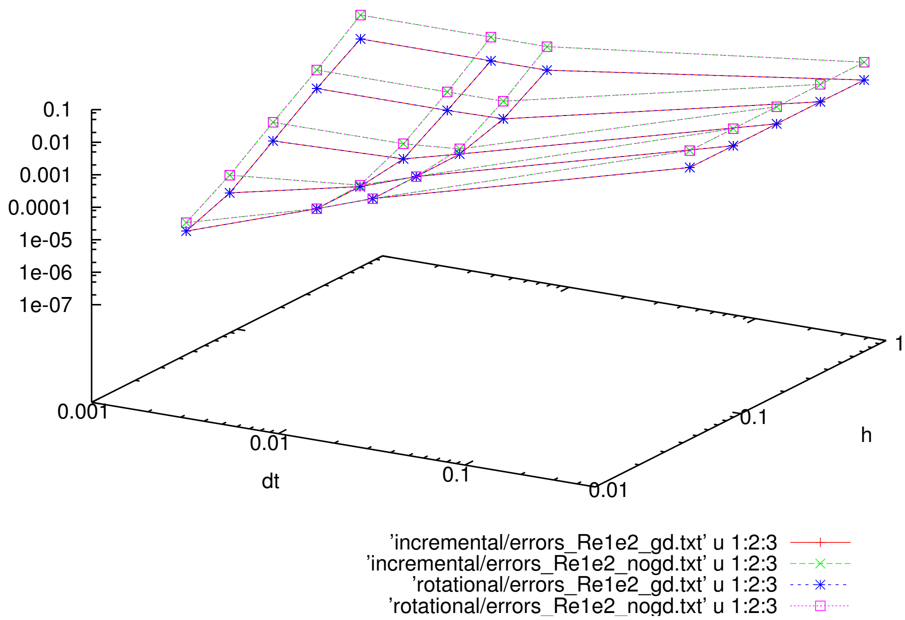

Figure 4.10: $Re = 10^2$, errors for the velocity w.r.t $L^2(\boldsymbol{u})$

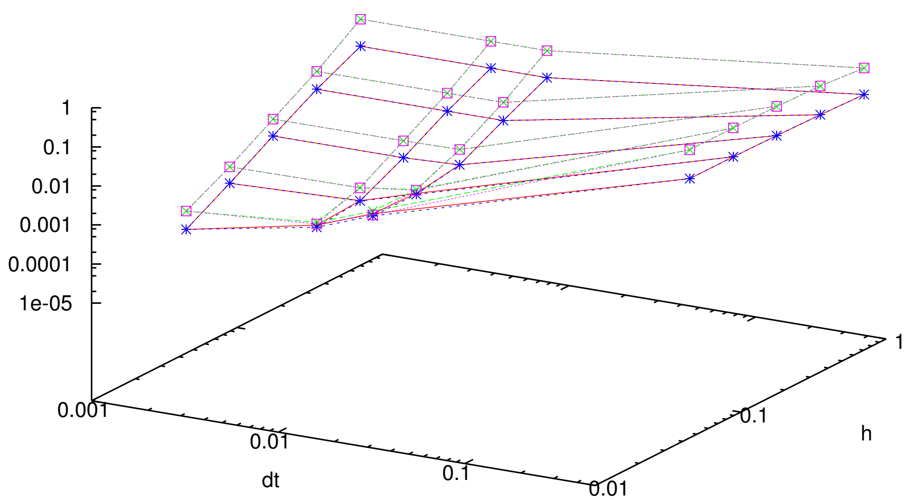

Figure 4.11: $Re = 10^2$, errors for the velocity w.r.t $H^1(\boldsymbol{u})$

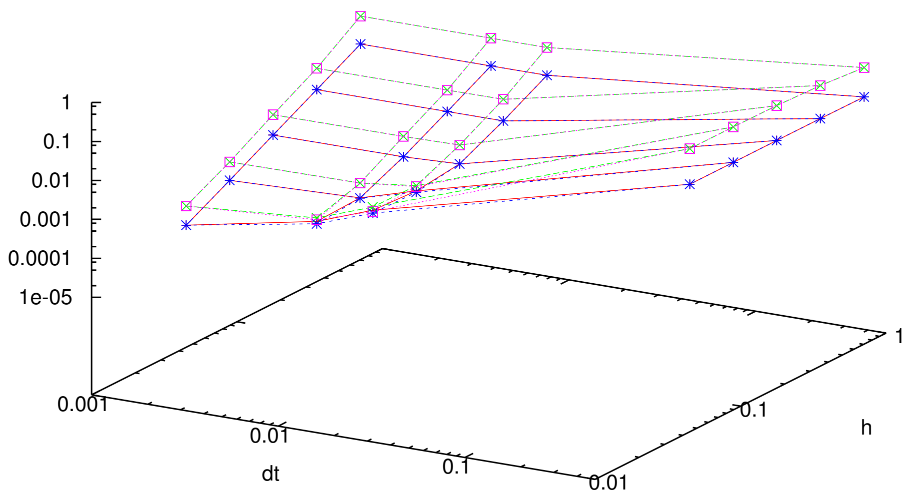

Figure 4.12: $Re = 10^2$, errors for the velocity w.r.t $L^2(\nabla \cdot \boldsymbol{u})$

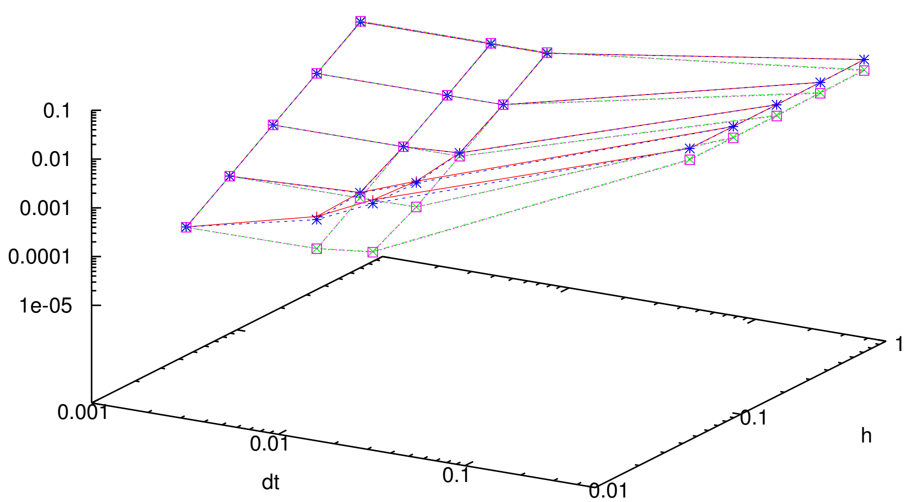

Figure 4.13: $Re = 10^2$, errors for the pressure w.r.t $L^2(p)$

Chapter 5

Discussion

We considered two different approaches for error estimates of the fully discretized time-dependent Navier-Stokes problem. In both cases we used a stabilized Finite Element framework for the spatially discretization and a BDF2-based projection algorithm for the discretization in time.

While we finally achieved similar results with respect to rates of convergence in space and time, the requirements are quite different.

In the first approach we were able to carry out the analysis under relatively mild regularity conditions on the intermediate solutions, i.e. we did not exceed regularity assumptions that stem from the definitions on the norms we wanted to control the error in. Unfortunately, this did not give the correct rates of convergence due to suboptimal estimates for the interpolation error $\widetilde{\boldsymbol{\eta}}_{u,h}$. However, assuming the same regularity in space as for the continuous problem lifted this restriction and with respect to the velocity just the estimate on the $L^2(\Omega)$ error is suboptimal. The pressure error always depends on the estimates for the velocity since we recover the $L^2(\Omega)$ error using the inf-sup stability of the considered ansatz spaces. The final result that we get is of kind

$$\Delta t \sum_{n=1}^{m} \|\boldsymbol{\zeta}_u^n\|_0^2 \leq C_{G,h} h^{2k_p+2} \Delta t + C_{G,h} \frac{h^{2k_u+2}}{\Delta t} + C_{G,h} h^{2k_u} + C_{G,t} (\Delta t)^4$$

$$\Delta t \sum_{n=1}^{m} \left(\nu \|\nabla \boldsymbol{\zeta}_u^n\|_0^2 + \gamma \|\nabla \cdot \boldsymbol{\zeta}_u^n\|_0^2 + \sum_{M \in \mathcal{M}_h} \tau_M^n \|\kappa_M((\widetilde{\boldsymbol{u}}_{ht}^n \cdot \nabla)\boldsymbol{\zeta}_u^n)\|_{0,M}^2 \right)$$
$$\leq C_{G,h} h^{2k_p+2} \Delta t + C_{G,h} \frac{h^{2k_u+2}}{\Delta t} + C_{G,t} (\Delta t)^2 + C_{G,h} h^{2k_u}$$

and

$$\Delta t \sum_{n=1}^{m} \|\zeta_p^{n-1}\|_0^2$$
$$\leq C_{G,h} \frac{h^{2k_p+2}}{\Delta t} + C_{G,h} \frac{h^{2k_u+2}}{(\Delta t)^3} + C_{G,h} \frac{h^{2k_u}}{(\Delta t)^2} + h^{2k_u} + C(\Delta t)^2.$$

That is we need to satisfy some compatibility constraints for the Δt and h depending on k_u and k_p due to these final estimates. Furthermore, we need to choose $\Delta t, h \le \nu^{3/2}$.

For the second approach we already needed to assume that the auxiliary solution is sufficiently smooth due to requirements for the spatially semi-discrete analysis. The final error is of form

$$\begin{aligned}
\|\boldsymbol{\zeta}_u\|^2_{l^\infty(t_0,T;[L^2(\Omega)]^d)} &\le C(\Delta t)^4 + Ch^{2k_u} \\
\|\boldsymbol{\zeta}_u\|^2_{l^2(t_0,T;LPS)} &\le C(\Delta t)^2 + Ch^{2k_u} \\
\|\zeta_p\|^2_{l^2(t_0,T;L^2(\Omega))} &\le \frac{\|\boldsymbol{\zeta}_u\|^2_{l^\infty(t_0,T;[L^2(\Omega)]^d)}}{(\Delta t)^2} + \|\boldsymbol{\zeta}_u\|^2_{l^2(t_0,T;LPS)} \\
&\le C\frac{(\Delta t)^4 + Ch^{2k_u}}{(\Delta t)^2} + C(\Delta t)^2 + Ch^{2k_u}.
\end{aligned}$$

In this estimate the errors with respect to spatial and discretization in time are separated for the velocity in contrast to our observations for the first approach. Nevertheless, the analysis dictates a choice due to $\Delta t \le \nu^3$ and the stability $h^{2k_u} \le \nu\Delta t \le \nu^4$. These requirements are clearly more restrictive than the previous ones although similar estimation techniques for the convective terms were used.

With respect to the stabilization parameters we obtained in both cases that $\gamma = \mathit{const.}$ should be chosen. While the first approach allows for a LPS stabilization parameter in the range $[0, 1/|\boldsymbol{u}_M|^2]$ in the second approach we need to consider the interval $[0, \min\left\{\frac{(\Delta t)^2}{|\widetilde{\boldsymbol{u}}_M^n|^2 h^{2(s-k_u)}}, h^d\right\}]$. Due to the fact that we often find a behavior like $h/|\boldsymbol{u}_M|$ this restriction seems to be too strict to allow for considerable effects.

Although the analytical results are convincing we still suffer from having too assume a quite a lot regularity on the intermediate solution. Furthermore, the interpolation rate of convergence for the $L^2(\Omega)$ norm can not be achieved in either of the approaches. We attempt to solve these problems by omitting the intermediate step and discretizing in space and time at the same time in future wirk. In particular, we need to find a sufficiently problem adapted interpolator such that the discretization error is superconvergent in space and time with respect to the LPS norm.

After these analytical observations we also investigated if we can see these effects also numerically. Therefore we compared with an algorithm that uses a rotational correction term. The results suggest that the error behaves like the interpolation error with respect to spatial discretization and converges like $(\Delta)^2$ in time for all considered norms. Especially, for low Reynolds number the rotational correction diminishes the velocity errors considerably. For higher Reynolds numbers the effect of grad-div stabilization is dominant. Interestingly, we observe for $Re = 10^2$ a positive effect for the velocity but a negative effect for the pressure. Thus, the choice of the stabilization parameter must depend on the norm one wants to control the error in.

Appendix A

Appendix

A.1 Splittings for the Discretized Time Derivative

In this report we need quite often a splitting for terms of the form

$$\langle 3a - 4b + c, a \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes a symmetric bilinear form.
Some auxiliary algebraic identities are:

$$\begin{aligned}
2\langle a, a-b \rangle &= |a|^2 + |a-b|^2 - |b|^2 \\
2\langle 3a - 4b + c, a \rangle &= |a|^2 + (4|a|^2 - 4\langle a, b \rangle + |b|^2) \\
&\quad + (|a|^2 + 4|b|^2 + |c|^2 - 4\langle a, b \rangle - 4\langle b, c \rangle + 2\langle a, c \rangle) \\
&\quad - |b|^2 - (4|b|^2 - 4\langle b, c \rangle + |c|^2) \\
&= |a|^2 + |2a - b|^2 + |a - 2b + c|^2 - |b|^2 - |2b - c|^2
\end{aligned}$$

where $|a|^2$ is an abbreviation for $\langle a, a \rangle$.
This gives for the desired term

$$\begin{aligned}
&2\langle 3a - 4b + c, a \rangle \\
&= 6\langle a - d, a \rangle + 2\langle 3d - 4b + c, a - d \rangle + 2\langle 3d - 4b + c, d \rangle \\
&= 3|a|^2 - 3|a - d|^2 - 3|d|^2 + 2\langle 3d - 4b + c, a - d \rangle \\
&\quad + |d|^2 + |2d - b|^2 + |d - 2b + c|^2 - |b|^2 - |2b - c|^2 \\
&= 3|a|^2 - 3|a - d|^2 - 2|d|^2 + 2\langle 3d - 4b + c, a - d \rangle \\
&\quad + |2d - b|^2 + |d - 2b + c|^2 - |b|^2 - |2b - c|^2.
\end{aligned}$$

Often we use this where a, b, c, d are given by

$$a = f(n), \quad b = g(n-1), \quad c = g(n-2), \quad d = g(n).$$

In this case we arrive at

$$
\begin{aligned}
&2\langle 3f(n) - 4g(n-1) + g(n-2), f(n)\rangle \\
&= 3|f(n)|^2 - 3|f(n) - g(n)|^2 - 2|g(n)|^2 + \\
&\quad + 2\langle 3g(n) - 4g(n-1) + g(n-2), f(n) - g(n)\rangle \\
&\quad + |2g(n) - g(n-1)|^2 + |\partial_{tt} g(n)|^2 \\
&\quad - |g(n-1)|^2 - |2g(n-1) - g(n-2)|^2
\end{aligned}
\tag{A.1.1}
$$

using the propagation operator $\delta_t f(n) := f(n) - f(n-1)$.

A.2 Vector Calculations

For a scalar function $\varphi \in H^1(\Omega)$ and a vector valued $\boldsymbol{u} \in H^3(\Omega)$ it holds via integration by parts

$$
\begin{aligned}
(\varphi, \nabla \cdot \Delta \boldsymbol{u}) &= \left(\varphi, \sum_{n=1}^{d} \frac{\partial (\Delta \boldsymbol{u})_i}{\partial x_i}\right) = \sum_{i,j} \left(\varphi, \frac{\partial^3 u_i}{\partial x_i \partial x_j^2}\right) \\
&= -\sum_{i,j} \left(\frac{\partial \varphi}{\partial x_j}, \frac{\partial^2 u_i}{\partial x_i \partial x_j}\right) = -\sum_{j} \left((\nabla \varphi)_j, \frac{\partial}{\partial x_j} (\nabla \cdot \boldsymbol{u})\right) \\
&= -(\nabla \varphi, \nabla \nabla \cdot \boldsymbol{u}).
\end{aligned}
\tag{A.2.1}
$$

For all $\boldsymbol{u} \in H^3(\Omega)$ and $\varphi \in L^2(\Omega)$ it holds

$$
\begin{aligned}
(\nabla \cdot \Delta \boldsymbol{u}, \varphi) &= \left(\sum_{j=1}^{d} \frac{\partial (\Delta \boldsymbol{u})_j}{\partial x_j}, \varphi\right) = \left(\sum_{i,j=1}^{d} \frac{\partial^3 \boldsymbol{u}_j}{\partial x_j \partial x_i^2}, \varphi\right) \\
&= \left(\sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i^2} \nabla \cdot \boldsymbol{u}, \varphi\right) = (\Delta \nabla \cdot \boldsymbol{u}, \varphi).
\end{aligned}
\tag{A.2.2}
$$

A.3 Estimates for the Convective Term

Denote by $\boldsymbol{u}_M$ the average over a cell $M \in \mathcal{M}$, i.e.

$$
\begin{aligned}
\boldsymbol{u}_M :=& \frac{1}{|M|} \int_M \boldsymbol{u} \, dx. \\
\Rightarrow \|\boldsymbol{u}_M\|_{0,M}^2 =& \int_M \|\boldsymbol{u}_M\|_0^2 dx = |M| \|\boldsymbol{u}_M\|_0^2 \le \frac{1}{|M|} \left(\int_M \|\boldsymbol{u}\| \, dx\right)^2 \\
\le& \int_M \|\boldsymbol{u}\|_0^2 \, dx \le \|\boldsymbol{u}\|_{0,M}^2.
\end{aligned}
$$

Then we get the following estimates for the streamline derivative:

$$
\begin{aligned}
&\|(\boldsymbol{a}_M \cdot \nabla)\boldsymbol{b}\|_{0,M}^2 = \sum_i \left\| \sum_j a_{M,j} \frac{\partial b_i}{\partial x_j} \right\|_{0,M}^2 \\
&\overset{\Delta}{\leq} \sum_i \left(\sum_j \left\| a_{M,j} \frac{\partial b_j}{\partial x_i} \right\|_{0,M} \right)^2 = \sum_i \left(\sum_j |a_{M,j}| \left\| \frac{\partial b_j}{\partial x_i} \right\|_{0,M} \right)^2 \\
&\overset{l_1-l_2}{\leq} \sum_i \left(\sum_j |a_{M,j}|^2 \right) \left(\sum_j \left\| \frac{\partial b_j}{\partial x_i} \right\|_{0,M}^2 \right) \leq \left(\sum_j |a_{M,j}|^2 \right) \sum_{i,j} \left\| \frac{\partial b_j}{\partial x_i} \right\|_{0,M}^2 \\
&= \|\boldsymbol{a}_M\|_{0,M}^2 \|\boldsymbol{b}\|_{1,M}^2 \leq \frac{1}{|M|} \|\boldsymbol{a}_M\|_{0,M}^2 \|\boldsymbol{b}\|_{1,M}^2
\end{aligned}
$$

Furthermore, we use that the convective term can be estimated as follows

$$
\begin{aligned}
c(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) &\leq C \begin{cases} \|\boldsymbol{u}\|_1 \|\boldsymbol{v}\|_1 \|\boldsymbol{w}\|_1 & \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in H_0^1(\Omega), \\ \|\boldsymbol{u}\|_2 \|\boldsymbol{v}\|_0 \|\boldsymbol{u}\|_1 & \forall \boldsymbol{u} \in H^2(\Omega) \cap H_0^1(\Omega), \boldsymbol{v}, \boldsymbol{w} \in H_0^1(\Omega), \\ \|\boldsymbol{u}\|_2 \|\boldsymbol{v}\|_1 \|\boldsymbol{w}\|_0 & \forall \boldsymbol{u} \in H^2(\Omega) \cap H_0^1(\Omega), \boldsymbol{v}, \boldsymbol{w} \in H_0^1(\Omega), \\ \|\boldsymbol{u}\|_1 \|\boldsymbol{v}\|_2 \|\boldsymbol{w}\|_0 & \forall \boldsymbol{v} \in H^2(\Omega) \cap H_0^1(\Omega), \boldsymbol{u}, \boldsymbol{w} \in H_0^1(\Omega) \end{cases} \\
c(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u}) &\leq C \|\boldsymbol{u}\|_0^{1/2} \|\boldsymbol{u}\|_1^{3/2} \|\boldsymbol{v}\|_1 \quad \forall \boldsymbol{u}, \boldsymbol{v} \in H_0^1(\Omega).
\end{aligned}
\tag{A.3.1}
$$

A.4 The Inverse Stokes Operator

For the considered ansatz spaces $\boldsymbol{V}$ and Q we define the (grad-div stabilized) inverse Stokes operator $S\colon \boldsymbol{V} \to \boldsymbol{V}$ as the solution $(S\boldsymbol{v}, r) \in \boldsymbol{V} \times Q$ of the problem

$$
\begin{aligned}
\nu(\nabla S\boldsymbol{v}, \nabla \boldsymbol{w}) - (r, \nabla \cdot \boldsymbol{w}) + \gamma(\nabla \cdot S\boldsymbol{v}, \nabla \cdot \boldsymbol{w}) &= (\boldsymbol{v}, \boldsymbol{w}) \quad && \forall \boldsymbol{w} \in \boldsymbol{V} \\
(\nabla \cdot S\boldsymbol{v}, q) &= 0 && \forall q \in Q \\
S\boldsymbol{v}|_{\partial\Omega} &= 0.
\end{aligned}
\tag{A.4.1}
$$

In particular, $S\boldsymbol{v}$ is weakly solenoidal. Furthermore, we define the induced semi-norm $|\cdot|_*$ by

$$
|\boldsymbol{v}|_*^2 := (\boldsymbol{v}, S\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}.
$$

The ansatz spaces $\boldsymbol{V}$ and Q may be discrete (Chapter 3) or not (Chapter 2). With this setting we can prove the following properties:

Lemma A.4.1. *S has the following properties for all $\epsilon > 0$:*

$$
\begin{aligned}
&\nu^2 \|\nabla S\boldsymbol{v}\|_0^2 + \nu |\boldsymbol{v}|_*^2 \leq \|\boldsymbol{v}\|_0^2, \\
&\quad |\boldsymbol{v}|_*^2 = \nu(\nabla S\boldsymbol{v}, \nabla \boldsymbol{v}) + \gamma(\nabla \cdot S\boldsymbol{v}, \nabla \cdot \boldsymbol{v}) \\
&\quad\quad \geq \left(1 - \left(\frac{2\nu + \gamma}{\nu}\right)^2 \frac{\epsilon}{4}\right) \|\boldsymbol{v}\|_0^2 - \frac{1}{\epsilon} \|\boldsymbol{v} - \boldsymbol{v}^*\|_0^2 \quad \forall \boldsymbol{v}^* \in \boldsymbol{V}^{div}
\end{aligned}
\tag{A.4.2}
$$

Proof. By testing equation (A.4.1) symmetrically we can derive an estimate on the solution in the H^1-semi-norm

$$\begin{aligned}\nu\|\nabla S\boldsymbol{v}\|_0^2 + \gamma\|\nabla\cdot S\boldsymbol{v}\|_0^2 &= (\boldsymbol{v}, S\boldsymbol{v}) - (r, \nabla\cdot S\boldsymbol{v}) = (\boldsymbol{v}, S\boldsymbol{v})\\ &\le \|\boldsymbol{v}\|_{-1}\|\nabla S\boldsymbol{v}\|\\ \Rightarrow \|\nabla S\boldsymbol{v}\| \le \frac{1}{\nu}\|\boldsymbol{v}\|_{-1} &\le \frac{1}{\nu}\|\boldsymbol{v}\|_0\end{aligned} \tag{A.4.3}$$

and get an estimate for the semi-norm induced by S with respect to the $L^2(\Omega)$ norm

$$|\boldsymbol{v}|_*^2 = (\boldsymbol{v}, S\boldsymbol{v}) \le \|\boldsymbol{v}\|_{-1}\|\nabla S\boldsymbol{v}\| \le \frac{1}{\nu}\|\boldsymbol{v}\|_{-1}^2 \le \frac{1}{\nu}\|\boldsymbol{v}\|_0^2. \tag{A.4.4}$$

According to the inf-sup condition (1.1.3), we have for $r \in Q$ the existence of a unique $\boldsymbol{w} \in \boldsymbol{V}$ with

$$\begin{aligned}&(\nabla\cdot\boldsymbol{w}, q) = -(r, q) \quad \forall q \in Q\\ &\|\nabla\boldsymbol{w}\| \le \beta^{-1}\|r\|.\end{aligned}$$

Testing with $(\boldsymbol{w}, 0) \in \boldsymbol{V} \times Q$, we obtain

$$\begin{aligned}\beta\|\nabla\boldsymbol{w}\|\|r\| \le \|r\|_0^2 &= -(r, \nabla\cdot\boldsymbol{w})\\ &= (\boldsymbol{v}, \boldsymbol{w}) - \nu(\nabla S\boldsymbol{v}, \nabla\boldsymbol{w}) - \gamma(\nabla\cdot S\boldsymbol{v}, \nabla\cdot\boldsymbol{w})\\ &\le (\|\boldsymbol{v}\|_{-1} + (\nu+\gamma)\|\nabla S\boldsymbol{v}\|)\|\nabla\boldsymbol{w}\|\\ &\le \left(2 + \frac{\gamma}{\nu}\right)\|\boldsymbol{v}\|_{-1}\|\nabla\boldsymbol{w}\|.\end{aligned} \tag{A.4.5}$$

A combination of these estimates states

$$\|r\| + \nu\|\nabla S\boldsymbol{v}\| \le C\left(1 + \frac{\gamma}{\nu}\right)\|\boldsymbol{v}\|_{-1}. \tag{A.4.6}$$

Provided the solution is sufficiently smooth we test with $(-\Delta S\boldsymbol{v}, -\Delta r)$ to get

$$\begin{aligned}&\nu\|\Delta S\boldsymbol{v}\|_0^2 + \gamma\|\nabla\nabla\cdot S\boldsymbol{v}\|_0^2 = \nu(\nabla\cdot\nabla S\boldsymbol{v}, \nabla\cdot\nabla S\boldsymbol{v}) + \gamma(\nabla\nabla\cdot S\boldsymbol{v}, \nabla\nabla\cdot S\boldsymbol{v})\\ &= -\nu(\nabla S\boldsymbol{v}, \nabla\Delta S\boldsymbol{v}) + (r, \nabla\cdot\Delta S\boldsymbol{v}) - \gamma(\nabla\cdot S\boldsymbol{v}, \nabla\cdot\Delta S\boldsymbol{v}) - (\nabla\cdot S\boldsymbol{v}, \Delta r)\\ &= -(\boldsymbol{v}, \Delta S\boldsymbol{v}) \le \|\boldsymbol{v}\|\|\Delta S\boldsymbol{v}\|\\ &\Rightarrow \|\Delta S\boldsymbol{v}\| \le \frac{1}{\nu}\|\boldsymbol{v}\|\end{aligned}$$

If we consider a discrete space $\boldsymbol{V} = \boldsymbol{V}_h$ this bound can also be obtained via estimates in $H_0^1(\Omega)$ (cf. [4]).

For the pressure we get by testing with $\boldsymbol{w} = \nabla r$

$$\begin{aligned}\|\nabla r\|_0^2 &= -(r, \nabla\cdot\nabla r)\\ &= -\nu(\nabla S\boldsymbol{v}, \nabla\nabla r) - \gamma(\nabla\cdot S\boldsymbol{v}, \nabla\cdot\nabla r) + (\boldsymbol{v}, \nabla r)\end{aligned}$$

$$
\begin{aligned}
&= \nu(\Delta S\boldsymbol{v}, \nabla r) + \gamma(\nabla\nabla \cdot S\boldsymbol{v}, \nabla r) + (\boldsymbol{v}, \nabla r) \\
&\leq (\nu\|\Delta S\boldsymbol{v}\| + \gamma\|\nabla\nabla \cdot S\boldsymbol{v}\| + \|\boldsymbol{v}\|)\|\nabla r\| \\
&\leq ((\nu + \gamma)\|\Delta S\boldsymbol{v}\| + \|\boldsymbol{v}\|)\|\nabla r\| \\
&\leq \left(2 + \frac{\gamma}{\nu}\right)\|\nabla r\| \\
\Rightarrow \|\nabla r\| &\leq \left(2 + \frac{\gamma}{\nu}\right)\|\boldsymbol{v}\|
\end{aligned}
$$

using the vector identity $\nabla \times \nabla \times \boldsymbol{v} = \nabla\nabla \cdot \boldsymbol{v} - \Delta\boldsymbol{v}$ and $(\nabla \times \nabla \times \boldsymbol{v}, \nabla\nabla \cdot \boldsymbol{v}) = 0$.

Next we are interested in a lower bound for the semi-norm induced by the Stokes operator.

$$
\begin{aligned}
|\boldsymbol{v}|_*^2 =& \nu(\nabla S\boldsymbol{v}, \nabla\boldsymbol{v}) + \gamma(\nabla \cdot S\boldsymbol{v}, \nabla \cdot \boldsymbol{v}) \\
=& \|\boldsymbol{v}\|_0^2 + (r, \nabla \cdot \boldsymbol{v}) \\
=& \|\boldsymbol{v}\|_0^2 - (\nabla r, \boldsymbol{v} - \boldsymbol{v}^*) \quad \boldsymbol{v}^* \in \boldsymbol{V}^{div} \\
\geq& \|\boldsymbol{v}\|_0^2 - \|\nabla r\|\|\boldsymbol{v} - \boldsymbol{v}^*\| \\
\geq& \left(1 - \left(\frac{2\nu + \gamma}{\nu}\right)^2 \frac{\epsilon}{4}\right)\|\boldsymbol{v}\|_0^2 - \frac{1}{\epsilon}\|\boldsymbol{v} - \boldsymbol{v}^*\|_0^2 \quad \forall\epsilon > 0
\end{aligned}
\tag{A.4.7}
$$

□

A.5 Discrete Gronwall Lemma

Lemma A.5.1. (Discrete Gronwall lemma). *Let y^n, h^n, g^n, f^n be nonnegative sequences satisfying for all $0 \leq m \leq [T/k]$*

$$
y^m + k\sum_{n=0}^{m} h^n \leq B + k\sum_{n=0}^{m}(g^n y^n + f^n) \quad \text{with } k\sum_{n=0}^{[T/k]} g^n \leq M.
$$

Assume $kg^n < 1$ and let $\sigma = \max_{0\leq n\leq[T/k]}(1 - kg^n)^{-1}$. Then for all $0 \leq m \leq [T/k]$ it holds

$$
y^m + k\sum_{n=1}^{m} h^n \leq \exp(\sigma M)\left(B + k\sum_{n=0}^{m} f^n\right). \tag{A.5.1}
$$

Proof. A proof of this result can be found in [14], for instance. □

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