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FEM with Local Projection Stabilization for Incompressible Flows in Rotating Frames

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Abstract

We consider conforming finite element (FE) approximations of the time-dependent, incompressible Navier-Stokes problem in rotating frames with inf-sup stable approximation of velocity and pressure. In case of high Reynolds numbers, a local projection stabilization (LPS) method is considered. In particular, the idea of streamline upwinding is combined with stabilization of the divergence-free constraint and a stabilization for the Coriolis term. For the arising nonlinear semidiscrete problem a stability and convergence analysis is given. The spatial analysis is an extension to our previous result in [1] for inertial frame of references to rotating ones. The convergence with respect to time extends results for the Stokes case [2] in inertial frames of references to rotating ones. Some numerical experiments complement the theoretical results. *July 29, 2015*

1. Introduction

We consider the time-dependent Navier-Stokes equations

$$\partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + 2\boldsymbol{\omega}\times\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in} \ (0,T)\times\Omega, \tag{1.1}$$

$$\nabla\cdot\boldsymbol{u} = 0 \quad \text{in} \ (0,T)\times\Omega, \tag{1.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{in} \ (0,T)\times\partial\Omega, \tag{1.3}$$

$$\boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0(\cdot) \quad \text{in} \ \Omega \tag{1.4}$$

in a bounded polyhedral domain $\Omega\subset\mathbb{R}^d$, $d\in\{2,3\}$. Here $\boldsymbol{u}\colon(0,T)\times\Omega\to\mathbb{R}^d$ and $p\colon(0,T)\times\Omega\to\mathbb{R}$ denote the unknown velocity and pressure fields for given viscosity $Ek>0$ and external forces $\boldsymbol{f}\in[L^2(0,T;L^2(\Omega))]^d$.
Defining U_{ref}, L_{ref} and ω_{ref} as characteristic velocity, length and angular velocity for the considered problem, we can derive critical non-dimensional parameters.

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These are the Ekman number Ek, the Rossby number Ro and the Reynolds number Re defined by

$$Re = \frac{U_{ref} L_{ref}}{\nu} \quad Ro = \frac{U_{ref}}{2L_{ref}\omega_{ref}} \quad Ek = \frac{\nu}{2L_{ref}^2 \omega_{ref}} = \frac{Ro}{Re}$$

Using these quantities a non-dimensional version of the Navier-Stokes equations is given by

$$\hat{\partial}_t \hat{\boldsymbol{u}} - Ek \hat{\Delta} \hat{\boldsymbol{u}} + Ro(\hat{\boldsymbol{u}} \cdot \hat{\nabla})\hat{\boldsymbol{u}} + 2\hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{u}} + \hat{\nabla}\hat{p} = \hat{\boldsymbol{f}} \quad \text{in } (0,\hat{T}) \times \hat{\Omega}, \tag{1.5}$$

$$\hat{\nabla} \cdot \hat{\boldsymbol{u}} = 0 \quad \text{in } (0,\hat{T}) \times \hat{\Omega}, \tag{1.6}$$

$$\hat{\boldsymbol{u}} = \boldsymbol{0} \quad \text{in } (0,\hat{T}) \times \partial\hat{\Omega}, \tag{1.7}$$

$$\hat{\boldsymbol{u}}(0,\cdot) = \hat{\boldsymbol{u}}_0(\cdot) \quad \text{in } \hat{\Omega}. \tag{1.8}$$

where we indicate non-dimensionality by hat ˆ. For the sake of simplicity we will omit the hats ˆ in the following.

In this paper, we consider stabilized finite element (FE) approximations of problem (1.5)-(1.8). In particular, inf-sup stable velocity-pressure FE pairs are chosen together with local projection stabilization (LPS). To our knowledge, there are not many results available in the literature, even in the case of a inertial frame of reference. The stationary case was considered in [3] under the strong condition of small data. A related LPS model has been considered in [4] for the stationary problem under a small data assumption. Some results for the time-dependent case can be found in [5] and [6] where LPS-based subgrid models of Smagorinsky type were considered.

For the linear Oseen problem, Matthies & Tobiska [7] provide a comprehensive overview regarding stabilized FE methods, in particular, in the case of LPS methods for inf-sup stable FE methods. (For a corresponding review and presentation of LPS methods with equal-order interpolation of velocity and pressure, we refer to [8].) In [7], the authors consider basically two variants of LPS methods: (1) stabilization of the streamline derivative $\boldsymbol{b} \cdot \nabla$ together with grad-div stabilization, and (2) stabilization of the full velocity gradient.

Here, we consider variant (1) for the time-dependent Navier-Stokes problem by extending the analysis in our paper [1] for the case of an inertial frame of reference. As in [7] we consider different cases:

(i) Methods of order k without compatibility condition:
For standard pairs $\boldsymbol{V}_h \times Q_h \subset \boldsymbol{V} \times Q := [W_0^{1,2}(\Omega)]^d \times L_0^2(\Omega)$ of conforming inf-sup stable velocity/pressure approximation of polynomial order $k/k-1$ with $k \in \mathbb{N} \setminus \{1\}$, one- and two-level variants of the LPS method are shown to be of order k in the standard norm in $\boldsymbol{V} \times Q$. In the case of high Reynolds numbers $Re_\Omega := \|\boldsymbol{u}\|_{L^\infty(\Omega)} \text{diam}(\Omega)/\nu = Ro_\Omega / Ek_\Omega$, the analysis requires a relatively mild restriction on the mesh Reynolds number $Re_M := \|\boldsymbol{u}\|_{L^\infty(M)} h_M/\nu$. In the one-level case, no enrichment of the discrete velocity space $\boldsymbol{V}_h$ is necessary (with

possible exception of discontinuous spaces Q_h). The analysis heavily relies on working in the subspace $\boldsymbol{V}_h^{div}$ of discretely divergence-free functions.

Inspired by the approach of Burman & Fernandez in [9] for edge stabilized FE methods (with equal-order discrete velocity-pressure) to problem (1.5)-(1.8), we can show that in case of $\boldsymbol{u} \in [L^\infty(0,T;W^{1,\infty}(\Omega))]^d$ the Gronwall constants depends on the norm in this space but not explicitly on the Reynolds number.

(ii) Methods of order k with compatibility condition:
In order to avoid the restriction on the mesh Reynolds number Re_M, we consider such pairs $\boldsymbol{V}_h \times Q_h$ of polynomial order $k/k-1$ with a special interpolation operator in the discrete velocity space. This interpolator exists if a certain (macro-)elementwise compatibility condition between the discrete velocities on the fine mesh and on the projection space is valid, see [10]. Unfortunatelyly, this interpolator is in general not applicable in $\boldsymbol{V}_h^{div}$.

We show that, in case of the mentioned compatibility condition, the restriction on the mesh Reynolds number can be avoided. In particular, for one-level methods this condition eventually requires an enrichment of the discrete velocity space. Moreover, a careful selection of the discrete pressure space is necessary.

(iii) Methods of order $k+1/2$:
Finally, as in [7] we discuss methods of order $k+1/2$ in the case of $Ek \leq Ch$. For one-level methods, this is accomplished by increasing the polynomial order of the discrete pressure in the setting of methods (ii).

For inf-sup stable (but not exactly divergence-free) pairs $\boldsymbol{V}_h \times Q_h$, the application of the so-called grad-div (or grad-div) stabilization is important. A critical issue is the design of the stabilization parameter set. As a rule of thumb a globally constant value $\gamma \equiv \gamma_M \sim 1$ always improves mass conservation but might be different from case to case. Even for the Stokes problem with $Re_\Omega = 0$, the results in [11] give no general result. On the other hand, for simplicial meshes it is shown in [12] that solutions with Taylor-Hood elements $[\mathbb{P}_k]^d/\mathbb{P}_{k-1},\ k \geq d$ converge with $\gamma \to \infty$ to the (pointwise divergence-free) Scott-Vogelius solution. We address the choice of the stabilization parameter in more detail in numerical experiments.

Outline of the paper: In Section 2 we introduce the LPS method for the time-dependent Navier-Stokes problem. Then, in Section 3, stability issues and well-posedness of the method are discussed. Methods of order k without enrichment are considered in Section 4.1, whereas methods of order k with enrichment are the subject of Section 4.2. Methods of order $k+\frac{1}{2}$ are addressed in Section 4.3. Some numerical results together with a critical discussion of the parameter design are given in Section 6.

2. LPS Method for the Navier-Stokes Problem

In this section, we describe the model problem and the spatial semidiscretization based on inf-sup stable interpolation of velocity and pressure together with local projection stablization.

2.1. Time-Dependent Navier-Stokes Problem

In the following, we will consider the usual Sobolev spaces $W^{m,p}(\Omega)$ with norm $\|\cdot\|_{W^{m,p}(\Omega)}, m \in \mathbb{N}_0, p \geq 1$. In particular, we have $L^p(\Omega) = W^{0,p}(\Omega)$. Moreover, the closed subspaces $W_0^{1,2}(\Omega)$, consisting of functions in $W^{1,2}(\Omega)$ with zero trace on $\partial\Omega$, and $L_0^2(\Omega)$, consisting of L^2-functions with zero mean in Ω, will be used. The inner product in $L^2(D)$ with $D \subseteq \Omega$ will be denoted by $(\cdot,\cdot)_D$. In case of $D = \Omega$ we omit the index.

The variational formulation of problem (1.5)-(1.8) reads:
Find $\boldsymbol{\mathcal{U}} = (\boldsymbol{u}, p) \colon (0,T) \to \boldsymbol{V} \times Q := [W_0^{1,2}(\Omega)]^d \times L_0^2(\Omega)$ such that

$$(\partial_t \boldsymbol{u}, \boldsymbol{v}) + A_G(\boldsymbol{u}; \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{V}}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{\mathcal{V}} = (\boldsymbol{v}, q) \in \boldsymbol{V} \times Q \tag{2.1}$$

with the Galerkin form

$$\begin{aligned} A_G(\boldsymbol{w}; \boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{V}}) := & \underbrace{Ek(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + (2\boldsymbol{\omega} \times \boldsymbol{u}, \boldsymbol{v}) - (p, \nabla \cdot \boldsymbol{v}) + (q, \nabla \cdot \boldsymbol{u})}_{=:a_G(\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{V}})} \\ & + \underbrace{\frac{Ro}{2}\big[((\boldsymbol{w} \cdot \nabla)\boldsymbol{u}, \boldsymbol{v}) - ((\boldsymbol{w} \cdot \nabla)\boldsymbol{v}, \boldsymbol{u})\big]}_{=c(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})} . \end{aligned} \tag{2.2}$$

The skew-symmetric form of the convective term c is chosen for conservation purposes. In this paper, we will assume that the velocity field $\boldsymbol{u}$ belongs to $[L^\infty(0,T; W^{1,\infty}(\Omega))]^d$ which ensures uniqueness of the solution.

2.2. Finite Element Spaces

For a simplex $T \in \mathcal{T}_h$ or a quadrilateral/hexahedron T in $\mathbb{R}^d$, let $\hat{T}$ be the reference unit simplex or the unit cube $(-1,1)^d$. The bijective reference mapping $F_T \colon \hat{T} \to T$ is affine for simplices and multi-linear for quadrilaterals/ hexahedra. Let $\hat{\mathbb{P}}_l$ and $\hat{\mathbb{Q}}_l$ with $l \in \mathbb{N}_0$ be the set of polynomials of degree $\leq l$ and of polynomials of degree $\leq l$ in each variable separately. Moreover, we set

$$\mathbb{R}_l(\hat{T}) := \begin{cases} \mathbb{P}_l(\hat{T}) & \text{on simplices } \hat{T} \\ \mathbb{Q}_l(\hat{T}) & \text{on quadrilaterals/hexahedra } \hat{T}. \end{cases}$$

Bubble-enriched spaces are

$$\mathbb{P}_l^+(\hat{T}) := \mathbb{P}_l(\hat{T}) + b_{\hat{T}} \cdot \mathbb{P}_{l-2}(\hat{T}), \quad \mathbb{Q}_l^+(\hat{T}) := \mathbb{Q}_l(\hat{T}) + \psi \cdot \operatorname{span}\{\hat{x}_i^{r-1},\, i = 1, \dots, d\}$$

with polynomial bubble function $b_{\hat{T}} := \prod_{i=0}^{d} \hat{\lambda}_i \in \hat{\mathbb{P}}_{d+1}$ on the reference simplex $\hat{T}$ with barycentric coordinates $\hat{\lambda}_i$ and with d-quadratic function $\psi(\hat{x}) := \prod_{i=1}^{d}(1 - \hat{x}_i^2)$ on the reference cube. Define

$$\begin{aligned} Y_{h,-l} &:= \{\boldsymbol{v}_h \in L^2(\Omega) : \ \boldsymbol{v}_h|_T \circ F_T \in \mathbb{R}_l(\hat{T}) \ \forall T \in \mathcal{T}_h\}, \\ Y_{h,l} &:= Y_{h,-l} \cap W^{1,2}(\Omega) \end{aligned}$$

and bubble-enriched spaces $Y^+_{h,\pm l}$ accordingly.

For convenience, we write $\boldsymbol{V}_h = \mathbb{R}_k$ instead of $\boldsymbol{V}_h = [Y_{h,k}]^d \cap V$ (with obvious modifications for $\mathbb{R}_k^+$) and $Q_h = \mathbb{R}_{\pm(k-1)}$ instead of $Q_h = Y_{h,\pm(k-1)} \cap Q$.

Assumption 2.1: *Let $\boldsymbol{V}_h \subseteq [Y_{h,k}]^d \cap \boldsymbol{V}$ and $Q_h \subseteq Y_{h,-k-1} \cap Q$ be FE spaces satisfying a discrete inf-sup-condition*

$$\inf_{q \in Q_h \setminus \{0\}} \sup_{\boldsymbol{v} \in \boldsymbol{V}_h \setminus \{\boldsymbol{0}\}} \frac{(\nabla \cdot \boldsymbol{v}, q)}{\|\nabla \boldsymbol{v}\|_0 \|q\|_0} \geq \beta > 0 \tag{2.3}$$

with a constant β independent on h.

$\mathcal{E}_h$ is the set of inner element faces $E \notin \partial\Omega$ of $\mathcal{T}_h$. We denote by h_E the diameter of the face $E \in \mathcal{E}_h$. For two cells T_E and T'_E shared by E let n_E be the unit normal vector pointing from T_E into T'_E. For piecewise smooth functions w_h, we denote by $[w_h]_E := (w_h|_{T_E})|_E - (w_h|_{T'_E})|_E$ the jump over the face E.

2.3. Local Projection Stabilization

For a Galerkin approximation of problem (2.1)-(2.2) on an admissible partition $\mathcal{T}_h$ of the polyhedral domain Ω, consider finite dimensional spaces $\boldsymbol{V}_h \times Q_h \subset \boldsymbol{V} \times Q$. Then, the semidiscretized problem reads: Find $\boldsymbol{\mathcal{U}}_h = (\boldsymbol{u}_h, p_h)\colon (0,T) \to \boldsymbol{V}_h \times Q_h$ such that for all $\boldsymbol{\mathcal{V}}_h = (\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times Q_h$:

$$(\partial_t \boldsymbol{u}_h, \boldsymbol{v}_h) + A_G(\boldsymbol{u}_h; \boldsymbol{\mathcal{U}}_h, \boldsymbol{\mathcal{V}}_h) = (\boldsymbol{f}, \boldsymbol{v}_h). \tag{2.4}$$

The semidiscrete Galerkin solution of problem (2.4) may suffer from spurious oscillations due to poor mass conservation, dominating advection or dominating rotation. The idea of local projection stabilization (LPS) methods is to separate discrete function spaces into small and large scales and to add stabilization terms only on small scales.

Let $\{\mathcal{M}_h\}$ be a family of shape-regular macro decompositions of Ω into d-simplices, quadrilaterals ($d = 2$) or hexahedra ($d = 3$). In the one-level LPS-approach, one has $\mathcal{M}_h = \mathcal{T}_h$. In the two-level LPS-approach, the decomposition $\mathcal{T}_h$ is derived from $\mathcal{M}_h$ by barycentric refinement of d-simplices or regular (dyadic) refinement of quadrilaterals and hexahedra. We denote by h_T and h_M the diameter of cells $T \in \mathcal{T}_h$ and $M \in \mathcal{M}_h$. It holds $h_T \leq h_M \leq Ch_T$ for all $T \subset M$ and $M \in \mathcal{M}_h$.

ASSUMPTION 2.2: *Let the FE space $\boldsymbol{Y}_{h,k}$ satisfy the local inverse inequality*

$$\|\nabla \boldsymbol{v}_h\|_{0,M} \leq C h_M^{-1} \|\boldsymbol{v}_h\|_{0,M} \quad \forall \boldsymbol{v}_h \in \boldsymbol{Y}_{h,k},\ M \in \mathcal{M}_h. \tag{2.5}$$

ASSUMPTION 2.3: *There are (quasi-)interpolation operators $j_u \colon \boldsymbol{V} \to \boldsymbol{V}_h$ and $j_p \colon Q \to Q_h$ such that for all $M \in \mathcal{M}_h$, for all $\boldsymbol{w} \in \boldsymbol{V} \cap [W^{l,2}(\Omega)]^d$ with $2 \leq l \leq k+1$:*

$$\|\boldsymbol{w} - j_u \boldsymbol{w}\|_{0,M} + h_M \|\nabla(\boldsymbol{w} - j_u \boldsymbol{w})\|_{0,M} \leq C h_M^l \|\boldsymbol{w}\|_{W^{l,2}(\omega_M)} \tag{2.6}$$

and for all $q \in Q \cap H^l(M)$ with $2 \leq l \leq k$:

$$\|q - j_p q\|_{0,M} + h_M \|\nabla(q - j_p q)\|_{0,M} \leq C h_M^l \|q\|_{W^{l,2}(\omega_M)}. \tag{2.7}$$

on a suitable patch $\omega_M \supset M$. Moreover, let

$$\|\boldsymbol{v} - j_u \boldsymbol{v}\|_{L^\infty(M)} \leq C h_M |\boldsymbol{v}|_{W^{1,\infty}(M)} \quad \forall \boldsymbol{v} \in [W^{1,\infty}(M)]^d.$$

Let $\boldsymbol{D_M} \subset [L^\infty(M)]^d$ denote a FE space on $M \in \mathcal{M}_h$ for $\boldsymbol{u}_h$. For each $M \in \mathcal{M}_h$, let $\pi_M \colon [L^2(M)]^d \to \boldsymbol{D_M}$ be the orthogonal L^2-projection. Moreover, we denote by $\kappa_M := id - \pi_M$ the so-called fluctuation operator.

ASSUMPTION 2.4: *The fluctuation operator $\kappa_M = id - \pi_M$ provides the approximation property (depending on $\boldsymbol{D_M}$ and $s \in \{0, \cdots, k\}$):*

$$\|\kappa_M \boldsymbol{w}\|_{0,M} \leq C h_M^l \|\boldsymbol{w}\|_{W^{l,2}(M)},\ \forall \boldsymbol{w} \in W^{l,2}(M),\ M \in \mathcal{M}_h,\ l = 0, \ldots, s. \tag{2.8}$$

A sufficient condition for Assumption 2.4 is $[\mathbb{P}_{s-1}]^d \subset \boldsymbol{D_M}$.
For each macro element $M \in \mathcal{M}_h$, let the elementwise averaged streamline direction $\boldsymbol{u}_M \in \mathbb{R}^d$ and the elementwise averaged angular velocity $\boldsymbol{\omega}_M \in \mathbb{R}^d$ be such that

$$\begin{aligned} |\boldsymbol{u}_M| &\leq C \|\boldsymbol{u}\|_{L^\infty(M)}, & \|\boldsymbol{u} - \boldsymbol{u}_M\|_{L^\infty(M)} &\leq C h_M |\boldsymbol{u}|_{W^{1,\infty}(M)} \\ |\boldsymbol{\omega}_M| &\leq C \|\boldsymbol{\omega}\|_{L^\infty(M)}, & \|\boldsymbol{\omega} - \boldsymbol{\omega}_M\|_{L^\infty(M)} &\leq C h_M |\boldsymbol{\omega}|_{W^{1,\infty}(M)}. \end{aligned} \tag{2.9}$$

One possible definition is

$$\boldsymbol{u}_M := \frac{1}{|M|} \int_M \boldsymbol{u}(x)\ dx, \qquad \boldsymbol{\omega}_M := \frac{1}{|M|} \int_M \boldsymbol{\omega}(x)\ dx. \tag{2.10}$$

The semidiscrete LPS model reads:
Find $\boldsymbol{\mathcal{U}}_h = (\boldsymbol{u}_h, p_h) \colon (0,T) \to \boldsymbol{V}_h \times Q_h$, such that for all $\boldsymbol{\mathcal{V}}_h = (\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times Q_h$:

$$\begin{aligned} (\boldsymbol{f}, \boldsymbol{v}_h) =& (\partial_t \boldsymbol{u}_h, \boldsymbol{v}_h) + A_G(\boldsymbol{u}_h; \boldsymbol{\mathcal{U}}_h, \boldsymbol{\mathcal{V}}_h) \\ &+ s_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + t_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + a_h(\boldsymbol{\omega}, \boldsymbol{u}_h, \boldsymbol{\omega}, \boldsymbol{v}_h) + i_h(p_h, q_h) \end{aligned} \tag{2.11}$$

with the streamline-upwind (SUPG)-type stabilization s_h, the grad-div (or grad-div) stabilization t_h, the Coriolis stabilization a_h and pressure jump stabilizations i_h according to

$$s_h(\boldsymbol{w}_h; \boldsymbol{u}, \boldsymbol{v}) := \sum_{M \in \mathcal{M}_h} \tau_M(\boldsymbol{w}_M)(\kappa_M((\boldsymbol{w}_M \cdot \nabla)\boldsymbol{u}), \kappa_M((\boldsymbol{w}_M \cdot \nabla)\boldsymbol{v}))_M \tag{2.12}$$

$$t_h(\boldsymbol{w}_h; \boldsymbol{u}, \boldsymbol{v}) := \sum_{M \in \mathcal{M}_h} \gamma_M(\boldsymbol{w}_M)(\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v})_M, \tag{2.13}$$

$$a_h(\boldsymbol{w}_h^1, \boldsymbol{u}_h, \boldsymbol{w}_h^2, \boldsymbol{v}_h) := \sum_{M \in \mathcal{M}_h} \alpha_M(\boldsymbol{w}_M)(\kappa(\boldsymbol{w}_M^1 \times \boldsymbol{u}_h), \kappa(\boldsymbol{w}_M^2 \times \boldsymbol{v}_h))_M, \tag{2.14}$$

$$i_h(p, q) := \sum_{E \in \partial M, M \in \mathcal{M}_h} \phi_E([p]_E, [q]_E)_E. \tag{2.15}$$

The set of stabilization parameters $\tau_M(\boldsymbol{u}_h)$, $\alpha_M(\boldsymbol{\omega})$, $\gamma_M(\boldsymbol{u}_M)$, and ϕ_E has to be determined later on. For reasons to be discussed later, we impose:

Assumption 2.5: *Assume that for all $M \in \mathcal{M}_h$:*

$$\begin{aligned} &0 \le \tau_M(\boldsymbol{u}_M) \le \frac{\tau_0}{|\boldsymbol{u}_M|^2}, \qquad && \gamma_0 \max_M h_M \le \gamma_M(\boldsymbol{u}_M) \le \gamma_0, \\ &0 \le \alpha_M(\boldsymbol{\omega}_M) \le \frac{\alpha_0}{|\boldsymbol{\omega}_M|^2 h_M^2} && \phi_E \ge 0. \end{aligned} \tag{2.16}$$

In case of $\boldsymbol{u}_M = \boldsymbol{0}$ we set $\tau_M(\boldsymbol{u}_M) = 0$.

3. Stability Analysis

In this section, we derive stability estimates for the discrete velocity and pressure fields. Moreover, the existence of the solution of the LPS problem (2.11) is shown.

3.1. Notation

For the analysis, let us define the mesh-dependent expression $|||\cdot|||_{LPS}$ for all $\boldsymbol{\mathcal{V}} = (\boldsymbol{v}, q) \in V \times Q$ by

$$|||\boldsymbol{\mathcal{V}}|||_{LPS}^2 := Ek\|\nabla \boldsymbol{v}\|_0^2 + s_h(\boldsymbol{u}_h; \boldsymbol{v}, \boldsymbol{v}) + a_h(\boldsymbol{\omega}, \boldsymbol{u}_h, \boldsymbol{\omega}, \boldsymbol{v}) + t_h(\boldsymbol{u}_h; \boldsymbol{v}, \boldsymbol{v}) + i_h(q, q). \tag{3.1}$$

This is motivated by symmetric testing $\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{U}}$ together with $\boldsymbol{w} = \boldsymbol{u}_h$ in (2.2)

$$\begin{aligned} |||\boldsymbol{\mathcal{V}}|||_{LPS}^2 = &A_G(\boldsymbol{u}_h; \boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{V}}) + s_h(\boldsymbol{u}_h; \boldsymbol{v}, \boldsymbol{v}) \\ &+ t_h(\boldsymbol{u}_h; \boldsymbol{v}, \boldsymbol{v}) + a_h(\boldsymbol{\omega}, \boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{v}) + i_h(q, q) \end{aligned}$$

due to the skew-symmetric form of the convective term. In the case $i_h \equiv 0$, we will write

$$|||\boldsymbol{v}_h|||_{LPS} := |||(\boldsymbol{v}_h, 0)|||_{LPS}. \tag{3.2}$$

One basic idea of the numerical analysis is to handle the discrete velocity and pressure separately since Assumption 2.1 implies that

$$\boldsymbol{V}_h^{div} := \{\boldsymbol{v}_h \in \boldsymbol{V}_h \mid (\nabla \cdot \boldsymbol{v}_h, q_h) = 0 \ \ \forall q_h \in Q_h\} \neq \{\boldsymbol{0}\}. \tag{3.3}$$

3.2. Velocity Estimates

The first result gives control of the kinetic energy and of the dissipation terms for the discrete velocity $\boldsymbol{u}_h \in \boldsymbol{V}_h^{div}$.

LEMMA 3.1: *Let $\boldsymbol{f} \in L^1(0,T;L^2(\Omega))$ and $\boldsymbol{u}_0 \in L^2(\Omega)$. For $0 \le t \le T$, we obtain*

$$\frac{1}{2}\|\boldsymbol{u}_h(t)\|^2_{L^2(\Omega)} + \int_0^t |||\boldsymbol{u}_h(\tau)|||^2_{LPS}\, d\tau \le \|\boldsymbol{u}_h(0)\|^2_{L^2(\Omega)} + \frac{3}{2}\|\boldsymbol{f}\|^2_{L^1(0,t;L^2(\Omega))}. \tag{3.4}$$

Proof: Symmetric testing with $\boldsymbol{V}_h = (\boldsymbol{v}_h, 0) \in \boldsymbol{V}_h^{div} \times Q_h$ provides

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_h\|^2_{L^2(\Omega)} + |||\boldsymbol{u}_h|||^2_{LPS} \\
&= (\partial_t \boldsymbol{u}_h, \boldsymbol{u}_h) + A_G(\boldsymbol{u}_h; \boldsymbol{\mathcal{U}}_h, \boldsymbol{\mathcal{U}}_h) + s_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{u}_h) \\
&\quad + a_h(\boldsymbol{\omega}, \boldsymbol{u}_h, \boldsymbol{\omega}, \boldsymbol{u}_h) + t_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{u}_h) \\
&= (\boldsymbol{f}, \boldsymbol{u}_h).
\end{aligned} \tag{3.5}$$

Estimate (3.5) gives

$$\|\boldsymbol{u}_h\|_{L^2(\Omega)} \frac{d}{dt}\|\boldsymbol{u}_h\|_{L^2(\Omega)} = \frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_h\|^2_{L^2(\Omega)} \le (\boldsymbol{f}, \boldsymbol{u}_h) \le \|\boldsymbol{f}\|_{L^2(\Omega)}\|\boldsymbol{u}_h\|_{L^2(\Omega)},$$

hence $\frac{d}{dt}\|\boldsymbol{u}_h\|_{L^2(\Omega)} \le \|\boldsymbol{f}\|_{L^2(\Omega)}$. Integrating in time leads to

$$\|\boldsymbol{u}_h(t)\|_{L^2(\Omega)} - \|\boldsymbol{u}_0\|_{L^2(\Omega)} = \int_0^t \frac{d}{d\tau}\|\boldsymbol{u}_h(\tau)\|_{L^2(\Omega)}\, d\tau \le \int_0^t \|\boldsymbol{f}(\tau)\|_{L^2(\Omega)} d\tau$$

which provides

$$\|\boldsymbol{u}_h(t)\|_{L^2(\Omega)} \le \|\boldsymbol{u}_0\|_{L^2(\Omega)} + \|\boldsymbol{f}\|_{L^1(0,t;L^2(\Omega))}. \tag{3.6}$$

We start again from (3.5), integrate in time, apply (3.6) and Young's inequality:

$$\begin{aligned}
\frac{1}{2}\|\boldsymbol{u}_h(t)\|^2_{L^2(\Omega)} + \int_0^t |||\boldsymbol{u}_h(\tau)|||^2_{LPS}\, d\tau &\le \frac{1}{2}\|\boldsymbol{u}_h(0)\|^2_{L^2(\Omega)} + \int_0^t (\boldsymbol{f}(\tau), \boldsymbol{u}_h(\tau))d\tau \\
&\le \|\boldsymbol{u}_h(0)\|^2_{L^2(\Omega)} + \frac{3}{2}\|\boldsymbol{f}\|^2_{L^1(0,t;L^2(\Omega))},
\end{aligned}$$

i.e., estimate (3.4) is valid. □

Now, we can prove an existence result for the discrete velocity.

COROLLARY 3.1: *There exists a discrete solution* $\boldsymbol{u}_h \colon [0,T] \to \boldsymbol{V}_h^{div}$ *of the semidiscrete LPS model (2.11).*

Proof: We look for a solution $\boldsymbol{u}_h : [0,T] \to \boldsymbol{V}_h^{div}$ of the semidiscrete problem

$$\begin{aligned}(\partial_t \boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) - A_G(\boldsymbol{u}_h; \boldsymbol{\mathcal{U}}_h, \boldsymbol{\mathcal{V}}_h) - s_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) \\ - a_h(\boldsymbol{\omega}, \boldsymbol{u}_h, \boldsymbol{\omega}, \boldsymbol{v}_h) - t_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h)\end{aligned} \tag{3.7}$$

with appropriate initial condition $\boldsymbol{u}_h(0) = \boldsymbol{u}_{0h}$. $\boldsymbol{V}_h^{div}$ is a finite-dimensional Banach space and the right hand side of (3.7) continuously depends on $(t, \boldsymbol{u}_h) \in [0,T] \times \boldsymbol{V}_h^{div}$. As a consequence of Lemma 3.1, each (potential) solution of (3.7) is bounded on $[0,T]$. This implies boundedness of the right hand side on $[0,T] \times \boldsymbol{V}_h^{div}$. Then the generalized Peano theorem is applicable. A local solution of (3.7) can be extended to $[0,T]$. □

Remark: A uniqueness result for the semidiscrete problem (3.7) is still open. However, if we assume Lipschitz continuity in time for $\boldsymbol{f}$, the Picard-Lindelöf theorem yields uniqueness of the solution.

3.3. Pressure Estimates

The existence of the discrete pressure $p_h \in Q_h$ is guaranteed via Assumption 2.1. Moreover, we obtain the following stability result.

COROLLARY 3.2: *Let* $\boldsymbol{u}_h : [0,T] \to \boldsymbol{V}_h^{div} \subset \boldsymbol{V}_h$ *be a solution of the Cauchy problem (3.7). For* $0 \le t \le T$ *we obtain for the discrete pressure* p_h*:*

$$\begin{aligned}&\|p_h\|_{L^1(0,t;L^2(\Omega))} \\ &\le \frac{1}{\beta}\Big[\|\boldsymbol{f}\|_{L^1(0,t;(\boldsymbol{V}_h)^*)} + \|\partial_t \boldsymbol{u}_h\|_{L^1(0,t;(\boldsymbol{V}_h)^*)} + K\int_0^t |||\boldsymbol{u}_h(\tau)|||_{LPS}\, d\tau\Big]\end{aligned}$$

where

$$\begin{aligned}K :=& \sqrt{Ek} + \frac{C_P Ro \|\boldsymbol{u}_h\|_{L^\infty(0,t;L^\infty(\Omega))} + C_P^2 \|\boldsymbol{\omega}\|_{L^\infty(0,t;L^\infty(\Omega))}}{\sqrt{Ek}} \\ &+ \max_M \sqrt{\tau_M \|\boldsymbol{u}_M\|_{L^\infty(0,t;L^\infty(M))} + d\gamma_M + C_P \alpha_M \|\boldsymbol{\omega}\|_{L^\infty(0,t;L^\infty(M))}}.\end{aligned}$$

Proof: According to the discrete inf-sup condition (2.3), see Assumption 2.1, we have for all $p_h \in Q_h$ the existence of a unique $\boldsymbol{v}_h \in \boldsymbol{V}_h$ with

$$\nabla \cdot \boldsymbol{v}_h = -p_h, \qquad \|\nabla \boldsymbol{v}_h\|_{L^2(\Omega)} \le \beta^{-1} \|p_h\|_{L^2(\Omega)}. \tag{3.8}$$

Testing in equation (2.11) with $(\boldsymbol{v}_h, \boldsymbol{0}) \in \boldsymbol{V}_h \times Q_h$, we obtain via Friedrich's inequality that

$$
\begin{aligned}
\|p_h\|^2_{L^2(\Omega)} \;&=\; -(p_h, \nabla\cdot \boldsymbol{v}_h)\\
&= (\boldsymbol{f}, \boldsymbol{v}_h) - (\partial_t \boldsymbol{u}_h, \boldsymbol{v}_h) - Ek(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) - 2(\boldsymbol{\omega}\times \boldsymbol{u}_h, \boldsymbol{v}_h) - c(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h)\\
&\quad - s_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) - t_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) - a_h(\boldsymbol{\omega}, \boldsymbol{u}_h, \boldsymbol{\omega}, \boldsymbol{v}_h)\\
&\leq \|\nabla \boldsymbol{v}_h\|_{L^2(\Omega)} \Big[\|\boldsymbol{f}\|_{(\boldsymbol{V}_h)^*} + \|\partial_t \boldsymbol{u}_h\|_{(\boldsymbol{V}_h)^*}\\
&\quad + \bigg(\sqrt{Ek} + \frac{C_P Ro \|\boldsymbol{u}_h\|_{L^\infty(\Omega)} + C_P^2 \|\boldsymbol{\omega}\|_{L^\infty(\Omega)}}{\sqrt{Ek}}\\
&\qquad + \max_M \sqrt{\tau_M \|\boldsymbol{u}_M\|_{L^\infty(M)} + d\gamma_M + C_P \alpha_M \|\boldsymbol{\omega}\|_{L^\infty(M)}} \bigg) |||\boldsymbol{u}_h|||_{LPS} \Big].
\end{aligned}
$$

From (3.8) we get

$$
\beta \|p_h\|_{L^2(\Omega)} \leq \|\boldsymbol{f}\|_{(\boldsymbol{V}_h)^*} + \|\partial_t \boldsymbol{u}_h\|_{(\boldsymbol{V}_h)^*} + K|||\boldsymbol{u}_h|||_{LPS}.
$$

Finally, the assertion follows via integration in time. □

4. Quasi-optimal Error Estimates

In this section, we derive quasi-optimal estimates for the kinetic energy and dissipation (including fluctuations terms). To this goal, we decompose the error:

$$
\boldsymbol{\mathcal{U}} - \boldsymbol{\mathcal{U}}_h = (\boldsymbol{\mathcal{U}} - J\boldsymbol{\mathcal{U}}) + (J\boldsymbol{\mathcal{U}} - \boldsymbol{\mathcal{U}}_h) =: \mathbf{A} + \boldsymbol{E}_h \equiv (\boldsymbol{\eta}_u, \eta_p) + (\boldsymbol{e}_h, r_h). \tag{4.1}
$$

Here, $J = (j_u, j_p)$ denotes an appropriate interpolator in $\boldsymbol{V}_h \times Q_h$. We are interested in methods of order k, i.e., there exists a constant $C > 0$, independent of critical data (like Ek, Ro and h) such that for $0 \leq t \leq T$ and a sufficiently smooth solution $(\boldsymbol{u}, p)$:

$$
\begin{aligned}
&\|\boldsymbol{e}_h\|^2_{L^\infty(0,t);L^2(\Omega))} + Ek\|\nabla \boldsymbol{e}_h\|^2_{L^2(0,t;W_0^{1,2}(\Omega)}\\
&\quad + \int_0^t \big[s_h(\boldsymbol{u}_h; \boldsymbol{e}_h, \boldsymbol{e}_h) + t_h(\boldsymbol{u}_h; \boldsymbol{e}_h, \boldsymbol{e}_h) + a_h(\boldsymbol{\omega}, \boldsymbol{e}_h, \boldsymbol{\omega}, \boldsymbol{e}_h) \big] d\tau \\
&\leq Ch^{2k} \Big(|\boldsymbol{u}|^2_{L^2(0,t;W^{k+1,2}(\Omega))} + \|\partial_t \boldsymbol{u}\|^2_{L^2(0,t;W^{k,2}(\Omega))} + |p|^2_{L^2(0,t;W^{k,2}(\Omega))} \Big).
\end{aligned} \tag{4.2}
$$

In a first step (see Subsec. 4.1), we prove this result for a wide range of FE pairs $\boldsymbol{V}_h \times Q_h$ under a (mild) mesh restriction. The basic tool will be to work in the space $\boldsymbol{V}_h^{div}$. In a second step (see Subsec. 4.2), we will remove the mesh restriction. This is accomplished under an additional inf-sup condition, see Assumption 4.2 below, which restricts the possible choices of $\boldsymbol{V}_h \times Q_h$. Finally, we want to identify methods of order $k + \frac{1}{2}$ for $Ek \leq Ch$, see Subsec. 4.3.

4.1. Methods of Order k without Compatibility Condition

We will perform the error analysis for the velocity $\boldsymbol{u}_h$ in $\boldsymbol{V}_h^{div}$. Following Girault & Scott [13], we apply a divergence-preserving interpolation $j_u : V \to \boldsymbol{V}_h^{div}$. It is shown in [13] that the approximation properties (2.6)-(2.7) in Assumption 2.3 remain valid on simplicial isotropic meshes if the right hand side Sobolev norms are taken on a patch $\omega_M \supset M$ and provided $k \geq d$. It is argued in [13] that the result can be easily extended to quadrilateral/hexahedral meshes and in this case to $k = 2, d = 3$.

We obtain the following quasi-optimal semidiscrete error estimate for the LPS-model (2.11) with vanishing pressure jump terms, i.e., with $i_h \equiv 0$.

THEOREM 4.1: *Let Assumption 2.1-2.5 be valid. Assume that* $\boldsymbol{u}_h(0) = j_u \boldsymbol{u}_0$. *If* $\boldsymbol{u} \in [L^\infty(0,T;W^{1,\infty}(\Omega))]^d$, *then we obtain for the discrete velocity approximation* $\boldsymbol{e}_h = \boldsymbol{u}_h - j_u \boldsymbol{u}$ *of the LPS-method (2.11):*

$$
\begin{aligned}
&\|\boldsymbol{e}_h\|^2_{L^\infty(0,t);L^2(\Omega))} + \int_0^t |||\boldsymbol{e}_h(\tau)|||^2_{LPS}\, d\tau \\
&\leq C \sum_M \int_0^t e^{C_G(\boldsymbol{u})(t-\tau)} \Big[(Ek + \tau_M |\boldsymbol{u}_M|^2 + \gamma_M d) \|\nabla \boldsymbol{\eta}_u(\tau)\|^2_{L^2(M)} \\
&\quad + \left(Ro \left(1 + \frac{h_M^2 \|\boldsymbol{u}\|^2_{L^\infty(M)}}{Ek} \right) h_M^{-2} + \alpha_M |\boldsymbol{\omega}_M|^2 \right) \|\boldsymbol{\eta}_u(\tau)\|^2_{L^2(M)} \\
&\quad + \|\partial_t \boldsymbol{\eta}_u(\tau)\|^2_{L^2(M)} + \tau_M |\boldsymbol{u}_M|^2 \|\kappa_M(\nabla \boldsymbol{u})(\tau)\|^2_{L^2(M)} \\
&\quad + \min\left(\frac{d}{Ek}; \frac{1}{\gamma_M} \right) \|\eta_p(\tau)\|^2_{L^2(M)} + \alpha_M |\boldsymbol{\omega}_M|^2 \|\kappa_M(\boldsymbol{u})\|^2_{L^2(M)} \Big] \, d\tau
\end{aligned}
\tag{4.3}
$$

with $(\boldsymbol{\eta}_u, \eta_p) = (\boldsymbol{u} - j_u \boldsymbol{u}, p - j_p p)$ *and the Gronwall constant*

$$
C_G(\boldsymbol{u}) = 1 + CRo|\boldsymbol{u}|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + ChRo\|\boldsymbol{u}\|^2_{L^\infty(0,T;W^{1,\infty}(\Omega))} \tag{4.4}
$$

where $h := \max_M h_M$.

Proof: Subtracting (2.11) from (2.1) with $\boldsymbol{\mathcal{V}}_h = (\boldsymbol{e}_h, 0) \in \boldsymbol{V}_h^{div} \times Q_h$ and using (4.1) leads to the error equation

$$
\begin{aligned}
0 &= (\partial_t(\boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{e}_h) + a_G(\boldsymbol{\mathcal{U}} - \boldsymbol{\mathcal{U}}_h, \boldsymbol{\mathcal{V}}_h) + c(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{e}_h) - c(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{e}_h) \\
&\quad - s_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{e}_h) - t_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{e}_h) - a_h(\boldsymbol{\omega}, \boldsymbol{u}_h, \boldsymbol{\omega}, \boldsymbol{e}_h) \\
&= (\partial_t \boldsymbol{\eta}_u, \boldsymbol{e}_h) + (\partial_t \boldsymbol{e}_h, \boldsymbol{e}_h) + a_G(\mathbf{A}, (\boldsymbol{e}_h, 0)) + a_G(\boldsymbol{E}_h, (\boldsymbol{e}_h, 0)) \\
&\quad + c(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{e}_h) - c(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{e}_h) + s_h(\boldsymbol{u}_h; \boldsymbol{e}_h, \boldsymbol{e}_h) - s_h(\boldsymbol{u}_h; j_u \boldsymbol{u}, \boldsymbol{e}_h) \\
&\quad + t_h(\boldsymbol{u}_h; \boldsymbol{e}_h, \boldsymbol{e}_h) - t_h(\boldsymbol{u}_h; j_u \boldsymbol{u}, \boldsymbol{e}_h) + a_h(\boldsymbol{\omega}, \boldsymbol{e}_h, \boldsymbol{\omega}, \boldsymbol{e}_h) - a_h(\boldsymbol{\omega}, j_u \boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{e}_h).
\end{aligned}
$$

Reordering the terms and using (3.1) implies

$$\begin{aligned}
&\frac{1}{2}\partial_t\|\boldsymbol{e}_h\|^2_{L^2(\Omega)} + |||\boldsymbol{e}_h|||^2_{LPS} \\
&= -(\partial_t\boldsymbol{\eta}_u, \boldsymbol{e}_h) - a_G(\mathbf{A}, (\boldsymbol{e}_h, 0)) + c(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{e}_h) - c(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{e}_h) \\
&\quad + s_h(\boldsymbol{u}_h; j_u\boldsymbol{u}, \boldsymbol{e}_h) + t_h(\boldsymbol{u}_h; j_u\boldsymbol{u}, \boldsymbol{e}_h) + a_h(\boldsymbol{\omega}, j_u\boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{e}_h) \\
&= -(\partial_t\boldsymbol{\eta}_u, \boldsymbol{e}_h) - Ek(\nabla\boldsymbol{\eta}_u, \nabla\boldsymbol{e}_h) - (2\boldsymbol{\omega}\times\boldsymbol{\eta}_u, \boldsymbol{e}_h) + (\eta_p, \nabla\cdot\boldsymbol{e}_h) \\
&\quad + c(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{e}_h) - c(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{e}_h) - s_h(\boldsymbol{u}_h; \boldsymbol{\eta}_u, \boldsymbol{e}_h) - t_h(\boldsymbol{u}_h; \boldsymbol{\eta}_u, \boldsymbol{e}_h) \\
&\quad + s_h(\boldsymbol{u}_h; \boldsymbol{u}, \boldsymbol{e}_h) - a_h(\boldsymbol{\omega}, \boldsymbol{\eta}, \boldsymbol{\omega}, \boldsymbol{e}_h) + a_h(\boldsymbol{\omega}, \boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{e}_h)
\end{aligned}$$

where we used $\nabla\cdot\boldsymbol{u} = 0$. Some of the right hand side terms can be bounded as follows:

$$\begin{aligned}
(\partial_t\boldsymbol{\eta}_u, \boldsymbol{e}_h) &\le \|\partial_t\boldsymbol{\eta}_u\|_{L^2(\Omega)}\|\boldsymbol{e}_h\|_{L^2(\Omega)} \;\le\; \frac{1}{2}\|\partial_t\boldsymbol{\eta}_u\|^2_{L^2(\Omega)} + \frac{1}{2}\|\boldsymbol{e}_h\|^2_{L^2(\Omega)} \\
Ek(\nabla\boldsymbol{\eta}_u, \nabla\boldsymbol{e}_h) &\le \sqrt{Ek}\|\nabla\boldsymbol{\eta}_u\|_{L^2(\Omega)}|||\boldsymbol{e}_h|||_{LPS}, \\
(\eta_p, \nabla\cdot\boldsymbol{e}_h) &\le \Big(\sum_M \min\Big(\frac{d}{Ek}; \frac{1}{\gamma_M}\Big)\|\eta_p\|^2_{L^2(M)}\Big)^{\frac{1}{2}}|||\boldsymbol{e}_h|||_{LPS}, \\
s_h(\boldsymbol{u}_h; \boldsymbol{\eta}_u, \boldsymbol{e}_h) &\le \Big(\sum_M \tau_M|\boldsymbol{u}_M|^2\|\nabla\boldsymbol{\eta}_u\|^2_{L^2(M)}\Big)^{\frac{1}{2}}|||\boldsymbol{e}_h|||_{LPS}, \\
t_h(\boldsymbol{u}_h; \boldsymbol{\eta}_u, \boldsymbol{e}_h) &\le \Big(\sum_M \gamma_M d\|\nabla\boldsymbol{\eta}_u\|^2_{L^2(M)}\Big)^{\frac{1}{2}}|||\boldsymbol{e}_h|||_{LPS} \\
s_h(\boldsymbol{u}_h; \boldsymbol{u}, \boldsymbol{e}_h) &\le \Big(\sum_M \tau_M|\boldsymbol{u}_M|^2\|\kappa_M(\nabla\boldsymbol{u})\|^2_{L^2(M)}\Big)^{\frac{1}{2}}|||\boldsymbol{e}_h|||_{LPS} \\
a_h(\boldsymbol{\omega}, \boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{e}_h) &\le \Big(\sum_M \alpha_M|\boldsymbol{\omega}_M|^2\|\kappa_M(\boldsymbol{u})\|^2_{L^2(M)}\Big)^{\frac{1}{2}}|||\boldsymbol{e}_h|||_{LPS} \\
a_h(\boldsymbol{\omega}, \boldsymbol{\eta}, \boldsymbol{\omega}, \boldsymbol{e}_h) &\le \Big(\sum_M \alpha_M|\boldsymbol{\omega}_M|^2\|\boldsymbol{\eta}_u\|^2_{L^2(M)}\Big)^{\frac{1}{2}}|||\boldsymbol{e}_h|||_{LPS}.
\end{aligned}$$

This implies

$$\begin{aligned}
&\frac{1}{2}\partial_t\|\boldsymbol{e}_h\|^2_{L^2(\Omega)} + |||\boldsymbol{e}_h|||^2_{LPS} \\
&\le \frac{1}{2}\|\partial_t\boldsymbol{\eta}_u\|^2_{L^2(\Omega)} + \frac{1}{2}\|\boldsymbol{e}_h\|^2_{L^2(\Omega)} + c(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{e}_h) - c(\boldsymbol{u};\boldsymbol{u},\boldsymbol{e}_h) \\
&\quad + |||\boldsymbol{e}_h|||_{LPS}\Big[\sqrt{Ek}\|\nabla\boldsymbol{\eta}_u\|_{L^2(\Omega)} + \Big(\sum_M \tau_M|\boldsymbol{u}_M|^2\|\nabla\boldsymbol{\eta}_u\|^2_{L^2(M)}\Big)^{\frac{1}{2}} \\
&\quad + \Big(\sum_M \gamma_M d\|\nabla\boldsymbol{\eta}_u\|^2_{L^2(M)}\Big)^{\frac{1}{2}} + \Big(\sum_M \min\Big(\frac{d}{Ek};\frac{1}{\gamma_M}\Big)\|\eta_p\|^2_{L^2(M)}\Big)^{\frac{1}{2}} \\
&\quad + \Big(\sum_M \tau_M|\boldsymbol{u}_M|^2\|\kappa_M(\nabla\boldsymbol{u})\|^2_{L^2(M)}\Big)^{\frac{1}{2}} + \Big(\sum_M \alpha_M|\boldsymbol{\omega}_M|^2\|\boldsymbol{\eta}_u\|^2_{L^2(M)}\Big)^{\frac{1}{2}} \\
&\quad + \Big(\sum_M \alpha_M|\boldsymbol{\omega}_M|^2\|\kappa_M(\boldsymbol{u})\|^2_{L^2(M)}\Big)^{\frac{1}{2}}\Big],
\end{aligned}$$

thus via Young's inequality

$$\begin{aligned}
&\frac{1}{2}\partial_t\|\boldsymbol{e}_h\|^2_{L^2(\Omega)} + (1-2\epsilon)|||\boldsymbol{e}_h|||^2_{LPS} \\
&\le \frac{1}{2}\|\partial_t\boldsymbol{\eta}_u\|^2_{L^2(\Omega)} + \frac{1}{2}\|\boldsymbol{e}_h\|^2_{L^2(\Omega)} + \big[c(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{e}_h) - c(\boldsymbol{u};\boldsymbol{u},\boldsymbol{e}_h)\big] \\
&\quad + \frac{7}{8\epsilon}\sum_M \Big[\big(Ek + \tau_M|\boldsymbol{u}_M|^2 + \gamma_M d\big)\|\nabla\boldsymbol{\eta}_u\|^2_{L^2(M)} \qquad (4.5) \\
&\quad + \min\Big(\frac{d}{Ek};\frac{1}{\gamma_M}\Big)\|\eta_p\|^2_{L^2(M)} + \tau_M|\boldsymbol{u}_M|^2\|\kappa_M(\nabla\boldsymbol{u})\|^2_{L^2(M)} \\
&\quad + \alpha_M|\boldsymbol{\omega}_M|^2\|\boldsymbol{\eta}_u\|^2_{L^2(M)} + \alpha_M|\boldsymbol{\omega}_M|^2\|\kappa_M(\boldsymbol{u})\|^2_{L^2(M)}\Big].
\end{aligned}$$

Lemma 7.1 in [1] yields for the convective terms:

$$\begin{aligned}
&(c(\boldsymbol{u};\boldsymbol{u},\boldsymbol{e}_h) - c(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{e}_h))/Ro \\
&\le \frac{1}{4\epsilon}\sum_M \frac{1}{h_M^2}\left(1 + \frac{h_M^2\|\boldsymbol{u}\|^2_{L^\infty(M)}}{Ek}\right)\|\boldsymbol{\eta}_u\|^2_{L^2(M)} + 3\epsilon|||\boldsymbol{\eta}_u|||^2_{LPS} + 4\epsilon|||\boldsymbol{e}_h|||^2_{LPS} \\
&\quad + \Big[\Big(|\boldsymbol{u}|_{W^{1,\infty}(\Omega)} + \Big(\epsilon h^2 + \frac{C}{\epsilon}\max_M \frac{h^2}{\gamma_M}\Big)|\boldsymbol{u}|^2_{W^{1,\infty}(\Omega)}\Big]\|\boldsymbol{e}_h\|^2_{L^2(\Omega)}
\end{aligned} \qquad (4.6)$$

assuming $\gamma_M \ge Ch_M$.

We summarize (4.5)-(4.6) and set $\epsilon = \frac{1}{12}$. Together with Assumption 2.5 we obtain

$$
\begin{aligned}
&\partial_t\|\boldsymbol{e}_h\|^2_{L^2(\Omega)} + |||\boldsymbol{e}_h|||^2_{LPS} \\
&\le \Big[1 + 2Ro|\boldsymbol{u}|_{W^{1,\infty}(\Omega)} + RoCh|\boldsymbol{u}|^2_{W^{1,\infty}(\Omega)}\Big]\|\boldsymbol{e}_h\|^2_{L^2(\Omega)} + \|\partial_t\boldsymbol{\eta}_u\|^2_{L^2(\Omega)} \\
&\quad + \sum_M \Big[\frac{43}{2}(Ek + \tau_M|\boldsymbol{u}_M|^2 + \gamma_M d)\|\nabla\boldsymbol{\eta}_u\|^2_{L^2(M)} \\
&\quad + 21\alpha_M|\boldsymbol{\omega}_M|^2\|\kappa_M(\boldsymbol{u})\|^2_{L^2(M)} \\
&\quad + \left(6\frac{Ro}{h_M^2}\left(1 + \frac{h_M^2\|\boldsymbol{u}\|^2_{L^\infty(M)}}{Ek}\right) + 21\alpha_M|\boldsymbol{\omega}_M|^2\|\right)\|\boldsymbol{\eta}_u\|^2_{L^2(M)} \\
&\quad + 21\tau_M|\boldsymbol{u}_M|^2\|\kappa_M(\nabla\boldsymbol{u})\|^2_{L^2(M)} + 21\min\left(\frac{d}{Ek};\frac{1}{\gamma_M}\right)\|\eta_p\|^2_{L^2(M)}\Big].
\end{aligned}
$$

Application of the Gronwall Lemma for $\|\boldsymbol{e}_h\|^2_{L^2(\Omega)}$ gives (4.3). Note that the initial error $\|\boldsymbol{e}_h(0)\|_{L^2(\Omega)}$ vanishes for $\boldsymbol{u}_h(0) = j_u\boldsymbol{u}_0$. □

Remark: The independence of the Gronwall constant $C_G(\boldsymbol{u})$ on the Reynolds number Re_Ω heavily relies on the lower bound $\gamma_M \ge Ch$ of the grad-div term parameter, together with the assumption $\boldsymbol{u} \in [L^\infty(0,T;W^{1,\infty}(\Omega)]^d$. The analysis uses at some places ideas of [9]. For $Ro = 0$ we are in the Stokes case and the Gronwall constant is independent of the Reynolds number no matter how γ is chosen.

Corollary 4.1: *Let Assumption 2.1-2.5 be valid and assume for the regularity of the smooth solutions $\boldsymbol{u} \in [L^\infty(0,T;W^{1,\infty}(\Omega))]^d$, $p \in L^2(0,T;Q)$ and $\partial_t\boldsymbol{u} \in [L^2(0,T;L^2(\Omega))]^d$. Then estimate (4.3) implies strong velocity convergence of the LPS-method in $[L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V)]^d$.*

Proof: For $\boldsymbol{u} \in L^2(0,T;V)$ and $p \in L^2(0,T;Q)$, a density argument gives

$$
\begin{aligned}
(Ek + \tau_M|\boldsymbol{u}_M|^2 + \gamma_M d)\|\nabla\boldsymbol{\eta}_u\|^2_{L^2(M)} &\to 0, \quad h_M \to 0, \\
\max\left(Ro; Ro\frac{\|\boldsymbol{u}\|^2_{L^\infty(M)}h_M^2}{Ek}; \alpha_M\|\boldsymbol{\omega}\|^2_{L^\infty(M)}h_M^2\right)\frac{1}{h_M^2}\|\boldsymbol{\eta}_u\|^2_{L^2(M)} &\to 0, \quad h_M \to 0, \\
\min\left(\frac{d}{Ek};\frac{1}{\gamma_M}\right)\|\eta_p\|^2_{L^2(M)} &\to 0, \quad h_M \to 0, \\
\|\partial_t\boldsymbol{\eta}_u|\|^2_{L^2(M)} &\to 0, \quad h_M \to 0, \\
\tau_M|\boldsymbol{u}_M|^2\|\kappa_M(\nabla\boldsymbol{u})\|^2_{L^2(M)} &\to 0, \quad h_M \to 0, \\
\alpha_M|\boldsymbol{\omega}_M|^2\|\kappa_M(\boldsymbol{u})\|^2_{L^2(M)} &\to 0, \quad h_M \to 0.
\end{aligned}
$$

Under the assumption of $\boldsymbol{u} \in [L^\infty(0,T;W^{1,\infty}(\Omega))]^d$ the exponent $C_G(\boldsymbol{u})$ of the Gronwall factor remains uniformly bounded for $h \to 0$. This fact and (4.3) imply strong convergence for $\boldsymbol{u}$ in $[L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V)]^d$. □

Corollary 4.2: *Let Assumption 2.1-2.5 be valid and assume a smooth solution of the time-dependent Navier-Stokes-problem according to*

$$\boldsymbol{u} \in [L^\infty(0,T;[W^{1,\infty}(\Omega)) \cap L^2(0,T;[W^{k+1,2}(\Omega))]^d,$$
$$\partial_t \boldsymbol{u} \in [L^2(0,T;W^{k,2}(\Omega))]^d,\ p \in L^2(0,T;W^{k,2}(\Omega)).$$

Set $\boldsymbol{u}_h(0) = j_u \boldsymbol{u}_0$ *and* $i_h \equiv 0$*. Then we obtain for* $0 \le t \le T$ *the semidiscrete a-priori estimate for the approximation* $\boldsymbol{e}_h = \boldsymbol{u}_h - j_u \boldsymbol{u}$ *of the LPS-method (2.11):*

$$\begin{aligned}
&\|\boldsymbol{e}_h\|^2_{L^\infty(0,t);L^2(\Omega))} + \int_0^t |||\boldsymbol{e}_h(\tau)|||^2_{LPS}\, d\tau \\
&\le C \sum_M h_M^{2k} \int_0^t e^{C_G(\boldsymbol{u})(t-\tau)} \Big[\min\big(\frac{d}{Ek}, \frac{1}{\gamma_M}\big) |p(\tau)|^2_{W^{k,2}(\omega_M)} \\
&\qquad + (Ek + Ro + \frac{Ro}{Ek} h_M^2 \|\boldsymbol{u}\|^2_{L^\infty(M)} + \tau_M |\boldsymbol{u}_M|^2 \\
&\qquad\quad + d\gamma_M + \alpha_M \|\boldsymbol{\omega}\|^2_{L^\infty(M)} h_M^2 \big) |\boldsymbol{u}(\tau)|^2_{W^{k+1,2}(\omega_M)} \\
&\qquad + \big(\tau_M |\boldsymbol{u}_M|^2 + \alpha_M h_M^2 \|\boldsymbol{\omega}\|^2_{L^\infty(M)}\big)\, h_M^{2(s-k)} |\boldsymbol{u}(\tau)|^2_{W^{s+1,2}(\omega_M)} \\
&\qquad + |\partial_t \boldsymbol{u}(\tau)|^2_{W^{k,2}(\omega_M)} \Big]\, d\tau
\end{aligned} \tag{4.7}$$

where $s \in \{0, \cdots, k\}$.

Proof: Interpolation results in $\boldsymbol{V}_h^{div} \times Q_h$ according to Assumption 2.3 provide

$$\begin{aligned}
&\sum_M (Ek + \tau_M |\boldsymbol{u}_M|^2 + d\gamma_M) \|\nabla \boldsymbol{\eta}_u(\tau)\|^2_{L^2(M)} + \sum_M \min\big(\frac{d}{Ek}, \frac{1}{\gamma_M}\big) \|\eta_p(\tau)\|^2_{L^2(M)} \\
&\quad + \sum_M \left((Ro + \frac{Ro}{Ek} h_M^2 \|\boldsymbol{u}\|^2_{L^\infty(M)}) h_M^{-2} + \alpha_M \|\boldsymbol{\omega}\|^2_{L^\infty(M)} \right) \|\boldsymbol{\eta}_u(\tau)\|^2_{L^2(M)} \\
&\le C \sum_M h_M^{2k} \Big(Ek + Ro + \frac{Ro}{Ek} h_M^2 \|\boldsymbol{u}\|^2_{L^\infty(M)} + \tau_M |\boldsymbol{u}_M|^2 \\
&\qquad + d\gamma_M + \alpha_M \|\boldsymbol{\omega}\|^2_{L^\infty(M)} h_M^2 \Big) |\boldsymbol{u}(\tau)|^2_{W^{k+1,2}(\omega_M)} \\
&\quad + C \sum_M h_M^{2k} \min\left(\frac{d}{Ek}; \frac{1}{\gamma_M}\right) \|p(\tau)\|^2_{W^{k,2}(\omega_M)}\ ds
\end{aligned}$$

and

$$\begin{aligned}
\|\partial_t \boldsymbol{\eta}_u\|^2_{L^2(\Omega)} &\le C \sum_M h_M^{2k} |\partial_t \boldsymbol{u}|^2_{W^{k,2}(\omega_M)}, \\
\tau_M |\boldsymbol{u}_M|^2 \|\kappa_M(\nabla \boldsymbol{u})\|^2_{L^2(M)} &\le C \sum_M \tau_M |\boldsymbol{u}_M|^2 h_M^{2s} |\boldsymbol{u}|^2_{W^{s+1,2}(\omega_M)}, \\
\alpha_M \|\boldsymbol{\omega}\|^2_{L^\infty(M)} \|\kappa_M(\boldsymbol{u})\|^2_{L^2(M)} &\le C \sum_M \alpha_M h_M^2 \|\boldsymbol{\omega}\|^2_{L^\infty(M)} h_M^{2s} |\boldsymbol{u}|^2_{W^{s+1,2}(\omega_M)}.
\end{aligned}$$

Using Assumption 2.5, this concludes the proof. □

Remark: The error estimate (4.7) does not blow up if

$$\frac{Ro}{Ek} h_M^2 \|\boldsymbol{u}\|^2_{L^\infty(M)} \le C \tag{4.8}$$

In dimensional form this condition corresponds to

$$\frac{Re_M^2}{Re} = \frac{h_M^2 \|\boldsymbol{u}\|^2_{L^\infty(M)}}{\nu^2} \frac{\nu}{L_{ref} U_{ref}} \le C \tag{4.9}$$

which gives a (mild) restriction on the local mesh width h_M. Thus we obtain a method of order k in the sense of (4.2) provided that $Re_M \le C/\sqrt{Re}$. In particular, there is no restriction if $Ro = 0$. □

Now we are in the position to derive bounds of the stabilization parameters. By formula (4.7) a possible choice for the set of stabilization parameters $(\tau_M)_M$, $(\gamma_M)_M$ and $(\alpha_M)_M$ is given by

$$\begin{aligned} 0 \le \tau_M(\boldsymbol{u}_M) &\le (Ek + Ro)\tau_0 \frac{h_M^{2(k-s)}}{|\boldsymbol{u}_M|^2}, \\ 0 \le \alpha_M(\boldsymbol{\omega}_M) &\le (Ro + Ek)\alpha_0 \frac{h_M^{2(k-s-1)}}{|\boldsymbol{\omega}_M|^2}, \\ (Ek + h)\gamma_0 \le \gamma_M &\le (Ek + Ro)\gamma_0 \end{aligned} \tag{4.10}$$

with $s \in \{0, 1, \dots, k\}$ and tuning constants $\tau_0, \gamma_0, \alpha_0 = \mathcal{O}(1)$. Let us remember the choice $\tau_M |\boldsymbol{b}_M|^2 \le C h_M^{k-s}$ and $\gamma \sim 1$ in [7], Table 1, for the Oseen problem.

A large range $0 \le \tau_M \le C(Ek + Ro) h_M^{2(k-s)} / |\boldsymbol{u}_M|^2$ is allowed, in particular $\tau_M \equiv 0$, thus showing a certain robustness of the grad-div stabilized Galerkin FEM with inf-sup stable interpolation. Nevertheless, the numerical experiments in Section 6 will show that the choice $s = k$ is appropriate (at least for boundary layer flows).

The approach of this subsection is applicable to almost all LPS-variants. We summarize possible variants of the triples $\boldsymbol{V}_h/Q_h/D_M$ with $t \in \{0, \dots, k-1\}$:

- One-level methods:
  $\mathbb{P}_k/\mathbb{P}_{k-1}/\mathbb{P}_t,\ \mathbb{Q}_k/\mathbb{Q}_{k-1}/\mathbb{Q}_t,\ \mathbb{P}_k^+/\mathbb{P}_{-(k-1)}/\mathbb{P}_t,\ \mathbb{Q}_k/\mathbb{P}_{-(k-1)}/\mathbb{P}_t$
- Two-level methods:
  $\mathbb{P}_k/\mathbb{P}_{k-1}/\mathbb{P}_t,\ \mathbb{Q}_k/\mathbb{Q}_{k-1}/\mathbb{Q}_t,\ \mathbb{P}_k^+/\mathbb{P}_{-(k-1)}/\mathbb{P}_t,\ \mathbb{Q}_k/\mathbb{P}_{-(k-1)}/\mathbb{P}_t$.

Remark: A-priori error estimate of the pressure can be derived following the lines in [14]. Unfortunately, one obtains an error reduction as a result of non-optimal estimates of $\partial_t \boldsymbol{e}_h$, see also [9].

4.2. Methods of Order k with Compatibility Condition

The restriction $\frac{Ro}{Ek} h_M^2 \|\boldsymbol{u}\|^2_{L^\infty(M)} \leq C$ in (4.8) stems from the estimate of the advective term in the analysis of Theorem 4.1. An improvement relies on the following

Assumption 4.2: *Let* $Y_{h,k}(M) = \{\boldsymbol{v}_h|_M \colon \ \boldsymbol{v}_h \in Y_{h,k}, \ \boldsymbol{v}_h = 0 \text{ on } \Omega \setminus M\}$ *and*

$$\exists \beta_u > 0 \colon \quad \inf_{\boldsymbol{w}_h \in D_M} \sup_{\boldsymbol{v}_h \in Y_{h,k}(M)} \frac{(\boldsymbol{v}_h, \boldsymbol{w}_h)_M}{\|\boldsymbol{v}_h\|_{L^2(M)} \|\boldsymbol{w}_h\|_{L^2(M)}} \geq \beta_u. \tag{4.11}$$

Lemma 4.1: *Let Assumption 4.2 be valid. Then there exists an interpolation operator* $i \colon \boldsymbol{V} \to \boldsymbol{V}_h$ *s.t. for* $1 \leq l \leq k+1$

$$\begin{gathered}(\boldsymbol{v} - i\boldsymbol{v}, \boldsymbol{w}_h) = 0 \qquad \forall \boldsymbol{w}_h \in D_h^u \ \forall \boldsymbol{v} \in \boldsymbol{V} \\ \|\boldsymbol{v} - i\boldsymbol{v}\|_{L^2(M)} + h_M |\boldsymbol{v} - i\boldsymbol{v}|_{W^{1,2}(M)} \leq C h_M^l \|\boldsymbol{v}\|_{W^{l,2}(\omega_M)} \ \forall \boldsymbol{v} \in \boldsymbol{V} \cap [W^{l,2}(\Omega)]^d.\end{gathered} \tag{4.12}$$

Proof: See Matthies et al. [10]. □

Condition (4.11) has two implications: At first, a careful selection of the discrete spaces $\boldsymbol{V}_h$ and D_M is required. Secondly, the interpolation operator $i \colon \boldsymbol{V} \to \boldsymbol{V}_h$ does not act in general in $\boldsymbol{V}_h^{div}$. As a consequence one has to modify the analysis of Theorem 4.1. In particular, a critical mixed term has to be handled. For discontinuous pressure space Q_h we have to include the pressure jump term i_h.

Theorem 4.3: *Let Assumption 2.1-4.2 be valid and assume for the regularity of the smooth solutions* $\boldsymbol{u} \in [L^\infty(0,T; W^{1,\infty}(\Omega))]^d$. *Moreover, consider a continuous or discontinuous discrete pressure space* $Q_h = \mathbb{P}_{k-1}$ *or* $Q_h = \mathbb{P}_{-(k-1)}$. *Then we obtain for* $0 \leq t \leq T$ *the error estimate*

$$\begin{aligned}
&\|\boldsymbol{e}_h\|^2_{L^\infty(0,t);L^2(\Omega))} + \int_0^t |||\boldsymbol{E}_h(\tau)|||^2_{LPS} \, d\tau \\
&\leq C \sum_M \int_0^t e^{C_G(\boldsymbol{u})(t-\tau)} \Big[(Ek + \tau_M |\boldsymbol{u}_M|^2 + d\gamma_M) \|\nabla \boldsymbol{\eta}_u(\tau)\|^2_{L^2(M)} \\
&\quad + \left(\frac{Ro}{h_M^2} + \frac{Ro}{\tau_M} + \alpha_M |\boldsymbol{\omega}_M|^2 \right) \|\boldsymbol{\eta}_u(\tau)\|^2_{L^2(M)} + \min\big(\frac{d}{Ek}, \frac{1}{\gamma_M}\big) \|\eta_p(\tau)\|^2_{L^2(M)} \\
&\quad + \tau_M |\boldsymbol{u}_M|^2 \|\kappa_M(\nabla \boldsymbol{u})(\tau)\|^2_{L^2(M)} + \alpha_M |\boldsymbol{\omega}_M|^2 \|\kappa_M(\boldsymbol{u})\|^2_{L^2(M)} \\
&\quad + \|\partial_t \boldsymbol{\eta}_u(\tau)\|^2_{L^2(M)} + \sum_{E \subset \partial M} \left(\frac{1}{\phi_E} \|\boldsymbol{\eta}_u(\tau) \cdot n_E\|^2_{L^2(E)} + \phi_E \|[\eta_p]\|^2_{L^2(E)} \right) \Big] \, d\tau
\end{aligned} \tag{4.13}$$

with $(\boldsymbol{\eta}_u, \eta_p) = (\boldsymbol{u} - i\boldsymbol{u}, p - j_p p)$ *and Gronwall constant*

$$\begin{aligned} C_G(\boldsymbol{u}) = 1 &+ CRo|\boldsymbol{u}|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + CRo\, h\|\boldsymbol{u}\|^2_{L^\infty(0,T;W^{1,\infty}(\Omega))} \\ &+ CRo \max_M \|\sqrt{\tau_M}\boldsymbol{u}_h\|^2_{L^\infty(0,T;W^{1,\infty}(M))} \end{aligned} \tag{4.14}$$

where $h := \max_M h_M$.

Proof: We modify the proof of Theorem 4.1. Eventually, the pressure jump stabilization term $i_h(\cdot,\cdot)$ is included, in particular in the expression $|||\cdot|||_{LPS}$. The first estimate (4.5) has to be modified as follows:

$$\begin{aligned} &\frac{1}{2}\partial_t\|\boldsymbol{e}_h\|^2_{L^2(\Omega)} + |||\boldsymbol{E}_h|||^2_{LPS} \\ &= -(\partial_t\boldsymbol{\eta}_u, \boldsymbol{e}_h) - Ek(\nabla\boldsymbol{\eta}_u, \nabla\boldsymbol{e}_h) - (2\boldsymbol{\omega}\times\boldsymbol{\eta}_u, \boldsymbol{e}_h) + (\eta_p, \nabla\cdot\boldsymbol{e}_h) - (r_h, \nabla\cdot\boldsymbol{\eta}_u) \\ &\quad + c(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{e}_h) - c(\boldsymbol{u};\boldsymbol{u},\boldsymbol{e}_h) - s_h(\boldsymbol{u}_h;\boldsymbol{\eta}_u,\boldsymbol{e}_h) - i_h(\eta_p, r_h) - t_h(\boldsymbol{u}_h;\boldsymbol{\eta}_u,\boldsymbol{e}_h) \\ &\quad + s_h(\boldsymbol{u}_h;\boldsymbol{u},\boldsymbol{e}_h) - a_h(\boldsymbol{\omega},\boldsymbol{\eta},\boldsymbol{\omega},\boldsymbol{e}_h) + a_h(\boldsymbol{\omega},\boldsymbol{u},\boldsymbol{\omega},\boldsymbol{e}_h). \end{aligned}$$

Note that the (critical) mixed term $(r_h, \nabla\cdot\boldsymbol{\eta}_u)$ does not vanish in general. Most of the right hand side terms can be bounded as in the proof of Theorem 4.1. The modifications due to Assumption 4.2 are as follows.

Lemma 7.2 in [1] provides a refined estimate of the advective error term:

$$\begin{aligned} &(c(\boldsymbol{u};\boldsymbol{u},\boldsymbol{e}_h) - c(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{e}_h))/Ro \\ &\le \frac{1}{2\epsilon}\sum_M \Big(\frac{1}{\tau_M} + \frac{1}{2h_M^2}\Big)\|\boldsymbol{\eta}_u\|^2_{L^2(M)} + 3\epsilon|||\boldsymbol{\eta}_u|||^2_{LPS} + 4\epsilon|||\boldsymbol{e}_h|||^2_{LPS} \\ &\quad + C\Big[|\boldsymbol{u}|_{W^{1,\infty}(\Omega)} + \max_M\Big(\Big(\epsilon h^2 + \epsilon\tau_M + \frac{1}{\epsilon}\max_M\frac{h^2}{\gamma_M}\Big)|\boldsymbol{u}|^2_{W^{1,\infty}(M)}\Big)\Big]\|\boldsymbol{e}_h\|^2_{L^2(\Omega)}. \end{aligned}$$

again assuming $\gamma_M \ge Ch_M$.
For the critical mixed error term, integration by parts gives

$$-(r_h, \nabla\cdot\boldsymbol{\eta}_u) = (\nabla r_h, \boldsymbol{\eta}_u) - \sum_{E\in\mathcal{E}_h}([r_h]_E, \boldsymbol{\eta}_u\cdot n_E)_E \tag{4.15}$$

Assume that $\nabla r_h|_M \in [\mathbb{P}_{k-2}(M)]^d$ which is possible for $Q_h = \mathbb{P}_{k-1}$ or $Q_h = \mathbb{P}_{-(k-1)}$. Then, the orthogonality condition of Lemma 4.1 is applicable, resulting in $(\nabla r_h, \boldsymbol{\eta}_u) = 0$. In case of continuous discrete pressure $Q_h = \mathbb{P}_{k-1}$, we have $[r_h]_E = 0$ and thus $(r_h, \nabla\cdot\boldsymbol{\eta}_u) = 0$. For discontinuous discrete pressure $Q_h = \mathbb{P}_{-(k-1)}$, we take advantage of the stabilization term j_h:

$$(\nabla r_h, \boldsymbol{\eta}_u) - \sum_{E\in\mathcal{E}_h}([r_h]_{0,E}, \boldsymbol{\eta}_u\cdot n_E)_E \le \Big(\sum_E \frac{1}{\phi_E}\|\boldsymbol{\eta}_u\cdot n_E\|^2_{L^2(E)}\Big)^{\frac{1}{2}}|||(\boldsymbol{e}_h, r_h)|||_{LPS}.$$

Moreover, we have

$$
\begin{aligned}
& s_h(\boldsymbol{u}_h; A, A) + t_h(\boldsymbol{u}_h; A, A) + i_h(A, A) \\
& \leq \sum_M (\tau_M |\boldsymbol{u}_M|^2 + d\gamma_M) \|\nabla \boldsymbol{\eta}_u\|^2_{L^2(M)} + \sum_E \phi_E \|[\eta_p]\|^2_{L^2(E)}.
\end{aligned}
$$

Summarizing all steps, we obtain the modified estimate

$$
\begin{aligned}
& \partial_t \|\boldsymbol{e}_h\|^2_{L^2(\Omega)} + |||\boldsymbol{E}_h|||^2_{LPS} \\
& \leq \Big[1 + 2|\boldsymbol{u}|_{W^{1,\infty}(\Omega)} + C(h + \max_M \tau_M)\|\boldsymbol{u}\|^2_{W^{1,\infty}(\Omega)}\Big] \|\boldsymbol{e}_h\|^2_{L^2(\Omega)} + \|\partial_t \boldsymbol{\eta}_u\|^2_{L^2(\Omega)} \\
& \quad + C \sum_M \Big[(Ek + \tau_M |\boldsymbol{u}_M|^2 + \gamma_M d)\|\nabla \boldsymbol{\eta}_u\|^2_{L^2(M)} + \alpha_M \|\boldsymbol{\omega}\|^2_{L^\infty(M)} \|\kappa_M(\boldsymbol{u})\|^2_{L^2(M)} \\
& \qquad + \left(\frac{Ro}{h_M^2} + \frac{Ro}{\tau_M} + \alpha_M \|\boldsymbol{\omega}\|^2_{L^\infty(M)}\right) \|\boldsymbol{\eta}_u\|^2_{L^2(M)} \\
& \qquad + \tau_M |\boldsymbol{u}_M|^2 \|\kappa_M(\nabla \boldsymbol{u})\|^2_{L^2(M)} + \min\left(\frac{d}{Ek}; \frac{1}{\gamma_M}\right) \|\eta_p\|^2_{L^2(M)}\Big] \\
& \quad + C \sum_E \left[\frac{1}{\phi_E} \|\boldsymbol{\eta}_u \cdot n_E\|^2_{L^2(E)} + \phi_E \|[\eta_p]\|^2_{L^2(E)}\right].
\end{aligned}
\tag{4.16}
$$

Application of the Gronwall Lemma for $\|\boldsymbol{e}_h\|^2_{L^2(\Omega)}$ gives (4.3). Note that the initial error $\|\boldsymbol{e}_h(0)\|_{L^2(\Omega)}$ vanishes for $\boldsymbol{u}_h(0) = j_u \boldsymbol{u}_0$. □

Finally, we have the following a-priori error estimate.

COROLLARY 4.3: *Let the assumptions of Theorem 4.3 be valid. Then we obtain*

$$
\begin{aligned}
& \|\boldsymbol{e}_h\|^2_{L^\infty(0,T);L^2(\Omega))} + \int_0^t |||\boldsymbol{E}_h(\tau)|||^2_{LPS} \, d\tau \\
& \leq C \sum_M h_M^{2k} \int_0^t e^{C_G(\boldsymbol{u})(t-\tau)} \Big[|\partial_t \boldsymbol{u}(\tau)|^2_{W^{k,2}(\omega_M)} \\
& \quad + \Big(\min\big(\frac{1}{Ek}, \frac{1}{\gamma_M}\big) + \frac{\phi_E}{h_E}\Big)|p(\tau)|^2_{W^{k,2}(\omega_M)} \\
& \quad + \big(Ro + Ek + \tau_M |\boldsymbol{u}_M|^2 + d\gamma_M \\
& \qquad + Ro\frac{h_M^2}{\tau_M} + \frac{h_E}{\phi_E} + \alpha_M h_M^2 \|\boldsymbol{\omega}\|^2_{L^\infty(M)}\Big) |\boldsymbol{u}(\tau)|^2_{W^{k+1,2}(\omega_M)} \\
& \quad + \big(\tau_M |\boldsymbol{u}_M|^2 + \alpha_M h_M^2 \|\boldsymbol{\omega}\|^2_{L^\infty(M)}\big) h_M^{2(s-k)} |\boldsymbol{u}(\tau)|^2_{W^{s+1,2}(\omega_M)}\Big] \, d\tau
\end{aligned}
\tag{4.17}
$$

From [7, 10] we obtain the following variants for $\boldsymbol{V}_h/Q_h/D_M$ for LPS-methods of order k with the crucial inf-sup condition (4.11) in Assumption 4.2:

- One-level methods:
  $\mathbb{P}_k^+/\mathbb{P}_{k-1}/\mathbb{P}_{k-1}$, $\mathbb{P}_k^+/\mathbb{P}_{-(k-1)}/\mathbb{P}_{k-1}$, $\mathbb{Q}_k/\mathbb{P}_{-(k-1)}/\mathbb{P}_{k-1}$

- Two-level methods:
  $\mathbb{P}_k/\mathbb{P}_{k-1}/\mathbb{P}_{k-1},\ \mathbb{Q}_k/\mathbb{Q}_{k-1}/\mathbb{P}_{k-1},\ \mathbb{P}_k^+/\mathbb{P}_{-(k-1)}/\mathbb{P}_{k-1},\ \mathbb{Q}_k/\mathbb{P}_{-(k-1)}/\mathbb{P}_{k-1}$,

thus giving the restriction $s = k$ for the projection space.

Remark: Regarding the stabilization parameters, we obtain from formula (4.17) that a method of order k results from

$$\begin{aligned} &\tau_0 h_M^2 \le \tau_M(\boldsymbol{u}_M) \le (Ro + Ek)\frac{\tau_0}{|\boldsymbol{u}_M|^2} && Ek\gamma_0 \le \gamma_M \le (Ek + Ro)\gamma_0 \\ &0 \le \alpha_M(\boldsymbol{\omega}_M) \le \frac{\alpha_0(Ro + Ek)}{h_M^2\|\boldsymbol{\omega}\|^2_{L^\infty(M)}} && \phi_0 h \le \phi_E \le \frac{\phi_0}{\gamma_M} h \end{aligned} \tag{4.18}$$

with tuning constants $\tau_0, \gamma_0, \alpha_0, \phi_0 = \mathcal{O}(1)$. The remarks in Subsec. 4.2 on the choice of γ_M remain valid. For discontinuous pressure spaces Q_h we may set $\phi_0 = \mathcal{O}(1)$, whereas $\phi_0 = 0$ for continuous pressure spaces Q_h. Note that in (4.18) τ_0 and γ_0 may still depend on $\boldsymbol{u}_M$. Moreover, a deterioration of the Gronwall constant $C_G(\boldsymbol{u})$ is not possible since, according to Assumption 2.5, we set $\tau_M(\boldsymbol{u}_M) = 0$ if $\boldsymbol{u}_M = 0$.

4.3. Methods of order $k + \frac{1}{2}$ with Compatibility Condition

The analysis of Subsec. 4.2 suggests to search for methods of order $k + \frac{1}{2}$ in the interesting case $Ek \le Ch$. As in [7] we will focus on one-level methods, i.e. $\mathcal{M}_h = \mathcal{T}_h$.

The definition of the LPS-scheme is the same as in Subsec. 4.2 with the exception of discrete pressure spaces Q_h of order k. From [7] we have the following variants for $\boldsymbol{V}_h/Q_h/D_M$ for LPS-methods with assumption Assumption 2.5:

- One-level methods:
  $\mathbb{P}_k^+/\mathbb{P}_k/\mathbb{P}_{k-1}\ \ (k \ge 1),\ \mathbb{P}_k^+/\mathbb{P}_{-k}/\mathbb{P}_{k-1}\ \ (k \ge d),\ \mathbb{Q}_k^+/\mathbb{P}_{-k}/\mathbb{P}_{k-1}\ \ (k \ge 2)$.

For brevity we give here only the final result. Please note that the result of Theorem 4.3 remains valid with the exception that the factor multiplying the seminorm $|\boldsymbol{u}(\tau)|^2_{W^{k+1,2}(\omega_M)}$ has to be replaced by $Ek + h_M + \tau_M|\boldsymbol{u}_M|^2 + d\gamma_M + Ro\frac{h_M^2}{\tau_M} + \frac{h_E}{\phi_E} + \alpha_M h_M^2\|\boldsymbol{\omega}\|^2_{L^\infty(M)}$.

Corollary 4.4: *Let Assumption 2.1-4.2 be valid and assume for the regularity of the smooth solutions* $\boldsymbol{u} \in [L^\infty(0,T;W^{1,\infty}(\Omega))]^d$. *Moreover, consider continuous or discontinuous discrete pressure* $Q_h = \mathbb{P}_k$ *or* $Q_h = \mathbb{P}_{-k}$. *Then, for*

$0 \le t \le T$, we obtain the error estimate

$$
\begin{aligned}
&\|\boldsymbol{e}_h\|^2_{L^\infty(0,t);L^2(\Omega))} + \int_0^t |||\boldsymbol{E}_h(\tau)|||^2_{LPS}\, d\tau \\
&\le C \sum_M h_M^{2k} \int_0^t e^{C_G(\boldsymbol{u})(t-\tau)} \Big[h_M^2 \min\big(\frac{1}{Ek}, \frac{1}{\gamma_M}\big) |p(\tau)|^2_{W^{k,2}(\omega_M)} \\
&\quad + h_M^2 |\partial_t \boldsymbol{u}(\tau)|^2_{W^{k+1,2}(\omega_M)} \\
&\quad + \big(Ek + h_M + \tau_M |\boldsymbol{u}_M|^2 + d\gamma_M \\
&\qquad + Ro\frac{h_M^2}{\tau_M} + \frac{h_E}{\phi_E} + \alpha_M h_M^2 \|\boldsymbol{\omega}\|^2_{L^\infty(M)} \big) |\boldsymbol{u}(\tau)|^2_{W^{k+1,2}(\omega_M)} \Big] d\tau
\end{aligned}
\tag{4.19}
$$

with the same Gronwall constant as in Theorem 4.3.

Remark: For the stabilization parameters (4.19) implies a modified design

$$
\begin{aligned}
\tau_M(\boldsymbol{u}_M) &= \tau_0 \min\Big(\frac{h_M}{|\boldsymbol{u}_M|^2}; Ro\frac{h_M^2}{Ek}\Big) \qquad && \gamma_M = \gamma_0 h_M \\
0 \le \alpha_M(\boldsymbol{\omega}_M) &\le \frac{\alpha_0}{h_M \|\boldsymbol{\omega}\|_{L^\infty(M)}} && \phi_E = \phi_0
\end{aligned}
\tag{4.20}
$$

with tuning constants $\alpha_0, \tau_0, \gamma_0 = \mathcal{O}(1)$. For discontinuous pressure spaces Q_h, we set again $\phi_0 = \mathcal{O}(1)$, whereas $\phi_0 = 0$ for continuous pressure spaces Q_h. In case of $Ek \le Ch$, this gives a method of order $k + \frac{1}{2}$ in the sense of [10].

From formula (4.19) in Subsec. 4.3 we obtain via equilibration the condition

$$
(Ek + \gamma_M)\|\boldsymbol{u}\|^2_{W^{k+1,2}(\omega_M)} \sim h_M^2 \min\big(\frac{1}{Ek}, \frac{1}{\gamma_M}\big) \|p\|^2_{W^{k,2}(\omega_M)}. \tag{4.21}
$$

In principle, the situation is as stated in Subsec. 4.1, but is crucially relaxed due to the factor h_T^2 on the right hand side. This motivates the choice (4.20).

5. Time Discretization

In order to fully discretize our model we use splitting method called rotational pressure-correction projection. This approach has been analyzed by Guermond and Shen in [2] is based on the backward differentiation formula of second order (BDF2). We define the operator D_t to abbreviate the discrete time derivative by

$$
D_t \boldsymbol{u}_{ht}^n := \frac{3\boldsymbol{u}_{ht}^n - 4\boldsymbol{u}_{ht}^{n-1} + \boldsymbol{u}_{ht}^{n-2}}{2\Delta t}. \tag{5.1}
$$

The fully discretized scheme then reads

$$
\begin{aligned}
&\Big(\frac{3\widetilde{\boldsymbol{u}}_{ht}^n - 4\boldsymbol{u}_{ht}^{n-1} + \boldsymbol{u}_{ht}^{n-2}}{2\Delta t}, \boldsymbol{v}_h \Big) + Ek(\nabla \widetilde{\boldsymbol{u}}_{ht}^n, \nabla \boldsymbol{v}_h) \\
&\quad + \frac{Ro}{2}\big[((\widetilde{\boldsymbol{u}}_{ht}^n \cdot \nabla)\widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) - ((\widetilde{\boldsymbol{u}}_{ht}^n \cdot \nabla)\boldsymbol{v}_h, \widetilde{\boldsymbol{u}}_{ht}^n)\big] + 2(\boldsymbol{\omega} \times \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) \\
&\quad + a_h(\boldsymbol{\omega}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{\omega}^n, \boldsymbol{v}_h) + s_h(\widetilde{\boldsymbol{u}}_{ht}^n; \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) + t_h(\widetilde{\boldsymbol{u}}_{ht}^n; \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) \\
&= (\boldsymbol{f}^n, \boldsymbol{v}_h) - (\nabla p_{ht}^{n-1}, \boldsymbol{v}_h)
\end{aligned}
\tag{5.2}
$$

$$\begin{aligned} &\left(\frac{3\boldsymbol{u}_{ht}^n - 3\widetilde{\boldsymbol{u}}_{ht}^n}{2\Delta t} + \nabla\phi_{ht}^n, \nabla q\right) = 0 \\ &p_{ht}^n = \phi_{ht}^n + p_{ht}^{n-1} - \pi_p(Ek\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^n) \end{aligned} \tag{5.3}$$

By eliminating the (weakly) solenoidal field $\widetilde{\boldsymbol{u}}_{ht}^n$ in (5.2) we arrive at the following scheme that is used in the implementation

$$\begin{aligned} &(D_t\widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) + Ek(\nabla\widetilde{\boldsymbol{u}}_{ht}^n, \nabla\boldsymbol{v}_h) + \frac{Ro}{2}\left[((\widetilde{\boldsymbol{u}}_{ht}^n\cdot\nabla)\widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) - ((\widetilde{\boldsymbol{u}}_{ht}^n\cdot\nabla)\boldsymbol{v}_h, \widetilde{\boldsymbol{u}}_{ht}^n)\right] \\ &+ 2(\boldsymbol{\omega}\times\widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) + a_h(\boldsymbol{\omega}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{\omega}^n, \boldsymbol{v}_h) \\ &+ s_h(\widetilde{\boldsymbol{u}}_{ht}^n; \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) + t_h(\widetilde{\boldsymbol{u}}_{ht}^n; \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) \\ &= (\boldsymbol{f}^n, \boldsymbol{v}_h) + \left(-\nabla p_{ht}^{n-1} - \frac{4}{3}\nabla\phi_{ht}^{n-1} + \frac{1}{3}\nabla\phi_{ht}^{n-2}, \boldsymbol{v}_h\right). \end{aligned} \tag{5.4}$$

5.1. Strategy and the Auxiliary Problem

Since we already have an estimate for the error induced by the spatial discretization we want to consider the error that appears when discretizing the spatial approximation in time. By the triangle inequality the total error is then bounded as

$$\|\boldsymbol{U} - \boldsymbol{U}_{ht}\| \leq \|\boldsymbol{U} - \boldsymbol{U}_h\| + \|\boldsymbol{U}_h - \boldsymbol{U}_{ht}\| \tag{5.5}$$

Since the convective term normally does not introduce any severe problems (according to Guermond) in the time discretization, we restrict the analysis at this point to the Stokes case, i.e. $Ro = 0$ and $\tau_M = 0$. Additionally, we assume that $\boldsymbol{\omega}$ is constant with respect to time. For the Navier-Stokes case in inertial frames of references we refer to [15].
The fully discretized quantities $\widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{u}_{ht}^n, p_{ht}^n$ solve the problem

$$\begin{aligned} &\left(\frac{3\widetilde{\boldsymbol{u}}_{ht}^n - 4\boldsymbol{u}_{ht}^{n-1} + \boldsymbol{u}_{ht}^{n-2}}{2\Delta t}, \boldsymbol{v}_h\right) + Ek(\nabla\widetilde{\boldsymbol{u}}_{ht}^n, \nabla\boldsymbol{v}_h) + 2(\boldsymbol{\omega}\times\widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) \\ &+ a_h(\boldsymbol{\omega}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{\omega}^n, \boldsymbol{v}_h) + t_h(\widetilde{\boldsymbol{u}}_{ht}^n; \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{v}_h) = (\boldsymbol{f}^n, \boldsymbol{v}_h) - (\nabla p_{ht}^{n-1}, \boldsymbol{v}_h) \end{aligned} \tag{5.6}$$

$$\begin{aligned} &\left(\frac{3\boldsymbol{u}_{ht}^n - 3\widetilde{\boldsymbol{u}}_{ht}^n}{2\Delta t} + \nabla\widetilde{\phi}_{ht}^n, \nabla q_h\right) = 0 \\ &p_{ht}^n = \widetilde{\phi}_{ht}^n + p_{ht}^{n-1} - \pi_p(Ek\nabla\cdot\boldsymbol{u}_{ht}^n) \end{aligned} \tag{5.7}$$

5.2. Initialization of the Auxiliary Problem

For initializing the algorithm we use a BDF1-scheme defined as follows

$$\begin{aligned} &\left(\frac{\widetilde{\boldsymbol{u}}_{ht}^1 - \boldsymbol{u}_h(0)}{\Delta t}, \boldsymbol{v}_h\right) + Ek(\nabla\widetilde{\boldsymbol{u}}_{ht}^1, \nabla\boldsymbol{v}_h) + 2(\boldsymbol{\omega}^1\times\widetilde{\boldsymbol{u}}_{ht}^1, \boldsymbol{v}_h) \\ &+ a_h(\boldsymbol{\omega}^1, \widetilde{\boldsymbol{u}}_{ht}^1, \boldsymbol{\omega}^1, \boldsymbol{v}_h) + t_h(\widetilde{\boldsymbol{u}}_{ht}^1, \boldsymbol{v}_h) = (-\nabla p_h(0) + \boldsymbol{f}^1, \boldsymbol{v}_h) \end{aligned} \tag{5.8}$$

$$\left(\frac{\boldsymbol{u}_{ht}^1 - \widetilde{\boldsymbol{u}}_{ht}^1}{\Delta t}, \nabla q_h\right) + (\nabla(p_{ht}^1 - p_h(0)), \nabla q_h) = 0. \tag{5.9}$$

Using the abbreviations $\boldsymbol{e}_u^1 := \boldsymbol{u}_h(t_1) - \boldsymbol{u}_{ht}^1$, $\widetilde{\boldsymbol{e}}_u^1 := \boldsymbol{u}_h(t_1) - \widetilde{\boldsymbol{u}}_{ht}^1$ and $e_p^1 := p_h(t_1) - p_{ht}^1$ we can state the estimates for the initial errors.

LEMMA 5.1: *The initial error due to time discretization can be bounded as*

$$\begin{aligned} \|\widetilde{\boldsymbol{e}}_u^1\|_0^2 + \|\widetilde{\boldsymbol{e}}_u^2\|_0^2 + \|\nabla \widetilde{\boldsymbol{e}}_u^1\|_0^2 + \|\nabla \widetilde{\boldsymbol{e}}_u^2\|_0^2 &\leq C(\Delta t)^4 \\ \|\nabla e_p^1\|_0^2 + \|\nabla e_p^2\|_0^2 &\leq C(\Delta t)^2 \end{aligned} \tag{5.10}$$

Proof: The error equation corresponding to (5.8) reads:

$$\begin{aligned} &\left(\frac{\widetilde{\boldsymbol{e}}^1}{\Delta t}, \boldsymbol{v}_h\right) + Ek(\nabla \widetilde{\boldsymbol{e}}^1, \nabla \boldsymbol{v}_h) + t_h(\widetilde{\boldsymbol{e}}^1, \boldsymbol{v}_h) + 2(\boldsymbol{\omega}^1 \times \widetilde{\boldsymbol{e}}_u^1, \boldsymbol{v}_h) \\ &+ a_h(\boldsymbol{\omega}^1, \widetilde{\boldsymbol{e}}_u^1, \boldsymbol{\omega}^1, \boldsymbol{v}_h) + (\nabla(p_h(t_1) - p_h(0)), \boldsymbol{v}_h) \\ &= \left(\frac{\boldsymbol{u}_h(t_1) - \boldsymbol{u}_h(0)}{\Delta t} - \partial_t \boldsymbol{u}_h(t_1), \boldsymbol{v}_h\right) =: (R_1, \boldsymbol{v}_h) \end{aligned} \tag{5.11}$$

Testing this equation with $\widetilde{\boldsymbol{e}}_u^1$ yields

$$\begin{aligned} &\|\widetilde{\boldsymbol{e}}_u^1\|_0^2 + Ek\Delta t \|\nabla \widetilde{\boldsymbol{e}}_u^1\|_0^2 + \gamma \Delta t \|\nabla \cdot \widetilde{\boldsymbol{e}}^1\|_0^2 + \Delta t \sum_M \left(\alpha_M \|\kappa_M(\boldsymbol{\omega}^1 \times \widetilde{\boldsymbol{e}}_u^1)\|_{0,M}^2\right) \\ &\leq \Delta t \min\{(\|(\nabla(p_h(0) - p_h(t_1))\|_0 + \|R_1\|_0)\|\widetilde{\boldsymbol{e}}_u^1\|_0, \\ &\qquad (\|p_h(0) - p_h(t_1)\|_0 + \|R_1\|_{-1})\|\nabla \widetilde{\boldsymbol{e}}_u^1\|_0\} \\ &\leq C(\Delta t)^2 \min\{\|\widetilde{\boldsymbol{e}}_u^1\|_0, \|\nabla \widetilde{\boldsymbol{e}}_u^1\|_0\} \end{aligned} \tag{5.12}$$

and hence $\|\widetilde{\boldsymbol{e}}_u^1 - \widetilde{\boldsymbol{e}}_u^0\|_0 = \|\widetilde{\boldsymbol{e}}_u^1\|_0 \leq C(\Delta t)^2$. Testing (5.11) with $\Delta \widetilde{\boldsymbol{e}}_u^1$ gives

$$\begin{aligned} &\|\nabla \widetilde{\boldsymbol{e}}_u^1\|_0^2 + Ek\Delta t \|\Delta \widetilde{\boldsymbol{e}}_u^1\|_0^2 + \gamma \Delta t \|\nabla \nabla \cdot \widetilde{\boldsymbol{e}}^1\|_0 + \Delta t \sum_M \left(\alpha_M \|\boldsymbol{\omega}_M^1 \times \nabla \widetilde{\boldsymbol{e}}_u^1\|_{0,M}^2\right) \\ &\leq \Delta t \min\{(\|(\Delta(p_h(0) - p_h(t_1))\|_0 + \|\nabla R_1\|_0)\|\nabla \widetilde{\boldsymbol{e}}_u^1\|_0, \\ &\qquad (\|(\nabla(p_h(0) - p_h(t_1))\|_0 + \|R_1\|_0)\|\Delta \widetilde{\boldsymbol{e}}_u^1\|_0\} \\ &\leq C(\Delta t)^2 \min\{\|\nabla \widetilde{\boldsymbol{e}}_u^1\|_0, \|\Delta \widetilde{\boldsymbol{e}}_u^1\|_0\} \end{aligned} \tag{5.13}$$

and this provides us with $\|\nabla \widetilde{\boldsymbol{e}}_u^1\|_0 \leq C(\Delta t)^2$, $\|\Delta \widetilde{\boldsymbol{e}}_u^1\|_0 \leq C\Delta t$.

Next we consider the error equation due to the projection step (5.9)

$$\left(\frac{\boldsymbol{e}_u^1 - \widetilde{\boldsymbol{e}}_u^1}{\Delta t}, \nabla q_h\right) + (\nabla(p_h(t_1) - p_{ht}^1), \nabla q_h) = (\nabla(p_h(t_1) - p_h(0)), \nabla q_h). \tag{5.14}$$

Choosing $q_h = p_h(t_1) - p_{ht}^1$ we arrive at

$$\Delta t\|\nabla(p_h(t_1) - p_{ht}^1)\|_0^2 \leq (\|\widetilde{\boldsymbol{e}}_u^1\|_0 + \Delta t\|(\nabla(p_h(0) - p_h(t_1))\|_0)\|\nabla(p_h(t_1) - p_{ht}^1)\|_0. \tag{5.15}$$

where we used that $\boldsymbol{e}^1$ is weakly solenoidal. Hence $\|\nabla(p_h(t_1) - p_{ht}^1)\|_0 \leq C\Delta t$ holds. Testing (5.14) with $\boldsymbol{e}_u^1$ gives

$$\|\boldsymbol{e}_u^1\|_0^2 \leq (\|\widetilde{\boldsymbol{e}}_u^1\|_0 + \Delta t\|(\nabla(p_h(0) - p_h(t_1))\|_0)\|\boldsymbol{e}_u^1\|_0 \tag{5.16}$$

and finally $\|\boldsymbol{e}_u^1\|_0 \leq C(\Delta t)^2$. Testing (5.14) with $-\Delta \boldsymbol{e}_u^1$ gives

$$\|\nabla \boldsymbol{e}_u^1\|_0 \leq (\|\nabla\widetilde{\boldsymbol{e}}_u^1\|_0 + \Delta t\|(\Delta(p_h(0) - p_h(t_1))\|_0) \leq C(\Delta t)^2. \tag{5.17}$$

Next, we need an estimate for $\widetilde{\boldsymbol{e}}_u^2$. Applying the same technique for $n = 2$ we get

$$\begin{aligned}
&(\frac{3\widetilde{\boldsymbol{e}}_u^2 - 3\widetilde{\boldsymbol{e}}_u^1}{2\Delta t}, \boldsymbol{v}_h) + Ek(\nabla(\widetilde{\boldsymbol{e}}_u^2 - \widetilde{\boldsymbol{e}}_u^1), \nabla \boldsymbol{v}_h) + a_h(\boldsymbol{\omega}^2, \widetilde{\boldsymbol{e}}_u^2 - \widetilde{\boldsymbol{e}}_u^1, \boldsymbol{\omega}^2, \boldsymbol{v}_h) \\
&\quad + t_h(\widetilde{\boldsymbol{e}}^2 - \widetilde{\boldsymbol{e}}^1, \boldsymbol{v}_h) + 2(\boldsymbol{\omega}^2 \times (\widetilde{\boldsymbol{e}}_u^2 - \widetilde{\boldsymbol{e}}_u^1), \boldsymbol{v}_h) \\
&= R^2 + \nabla(p_{ht}^1 - p_h(t_2), \boldsymbol{v}_h) - Ek(\nabla(\widetilde{\boldsymbol{e}}_u^1 - \widetilde{\boldsymbol{e}}_u^0), \nabla \boldsymbol{v}_h) + (\frac{3\boldsymbol{e}_u^1 - 3\widetilde{\boldsymbol{e}}_u^1}{2\Delta t}, \boldsymbol{v}_h) \\
&\quad + (\frac{\boldsymbol{e}_u^1 - \boldsymbol{e}_u^0}{2\Delta t}, \boldsymbol{v}_h) + a_h(\boldsymbol{\omega}^2, \widetilde{\boldsymbol{e}}_u^2 - \widetilde{\boldsymbol{e}}_u^1, \boldsymbol{\omega}^2, \boldsymbol{v}_h) + 2((\boldsymbol{\omega}^1 - \boldsymbol{\omega}^2) \times \widetilde{\boldsymbol{e}}_u^1, \boldsymbol{v}_h) \\
&= R^2 - Ek(\nabla(\widetilde{\boldsymbol{e}}_u^1 - \widetilde{\boldsymbol{e}}_u^0), \nabla \boldsymbol{v}_h) + (\frac{\boldsymbol{e}_u^1 - \boldsymbol{e}_u^0}{2\Delta t}, \boldsymbol{v}_h) + (\frac{3}{2}(\nabla(p_{ht}^1 - p_{ht}^0), \boldsymbol{v}_h) \\
&\quad + \nabla(p_{ht}^1 - p_h(t_2), \boldsymbol{v}_h) + a_h(\boldsymbol{\omega}^2, \widetilde{\boldsymbol{e}}_u^2 - \widetilde{\boldsymbol{e}}_u^1, \boldsymbol{\omega}^2, \boldsymbol{v}_h) + 2((\boldsymbol{\omega}^1 - \boldsymbol{\omega}^2) \times \widetilde{\boldsymbol{e}}_u^1, \boldsymbol{v}_h) \\
&= R^2 - Ek(\nabla(\widetilde{\boldsymbol{e}}_u^1 - \widetilde{\boldsymbol{e}}_u^0), \nabla \boldsymbol{v}_h) + (\frac{\boldsymbol{e}_u^1 - \boldsymbol{e}_u^0}{2\Delta t}, \boldsymbol{v}_h) + (\nabla(\frac{5}{2}p_{ht}^1 - \frac{3}{2}p_{ht}^0 - p_h(t_2)), \boldsymbol{v}_h) \\
&\quad + a_h(\boldsymbol{\omega}^2, \widetilde{\boldsymbol{e}}_u^2 - \widetilde{\boldsymbol{e}}_u^1, \boldsymbol{\omega}^2, \boldsymbol{v}_h) + 2((\boldsymbol{\omega}^1 - \boldsymbol{\omega}^2) \times \widetilde{\boldsymbol{e}}_u^1, \boldsymbol{v}_h) \\
&= R^2 - Ek(\nabla(\widetilde{\boldsymbol{e}}_u^1 - \widetilde{\boldsymbol{e}}_u^0), \nabla \boldsymbol{v}_h) + (\frac{\boldsymbol{e}_u^1 - \boldsymbol{e}_u^0}{2\Delta t}, \boldsymbol{v}_h) + (\nabla(\frac{5}{2}(p_{ht}^1 - p_h(t_1)), \boldsymbol{v}_h) \\
&\quad + (\nabla(\frac{5}{2}p_h(t_1) - \frac{3}{2}p_h(t_0) - p_h(t_2)), \boldsymbol{v}_h) + a_h(\boldsymbol{\omega}^2, \widetilde{\boldsymbol{e}}_u^2 - \widetilde{\boldsymbol{e}}_u^1, \boldsymbol{\omega}^2, \boldsymbol{v}_h) \\
&\quad + 2((\boldsymbol{\omega}^1 - \boldsymbol{\omega}^2) \times \widetilde{\boldsymbol{e}}_u^1, \boldsymbol{v}_h) \\
&\leq C\Delta t \min\{\|\boldsymbol{v}_h\|_0, \|\nabla \boldsymbol{v}_h\|_0\} \\
&\Rightarrow \|\widetilde{\boldsymbol{e}}_u^2 - \widetilde{\boldsymbol{e}}_u^1\|_0 \leq C(\Delta t)^2 \qquad Ek\|\widetilde{\boldsymbol{e}}_u^2 - \widetilde{\boldsymbol{e}}_u^1\|_1 \leq C(\Delta t)
\end{aligned} \tag{5.18}$$

Similarly as above we derive for the pressure error

$$\begin{aligned}
\Delta t\|\nabla(p_h(t_2) - p_{ht}^2)\|_0^2 &\leq (\|\widetilde{\boldsymbol{e}}_u^2\|_0 + \Delta t\|(\nabla(p_{ht}^1 - p_h(t_2))\|_0)\|\nabla(p_h(t_2) - p_{ht}^2)\|_0 \\
&\leq C(\Delta t)^2\|\nabla(p_h(t_2) - p_{ht}^2)\|_0
\end{aligned}$$

and therefore $\|\nabla e_p^2\| \leq C(\Delta t)$. □

5.3. Velocity Projection Error

For the following estimates we define the errors

$$\boldsymbol{e}_u^n := \boldsymbol{u}_h(t_n) - \boldsymbol{u}_{ht}^n \qquad \widetilde{\boldsymbol{e}}_u^n := \boldsymbol{u}_h(t_n) - \widetilde{\boldsymbol{u}}_{ht}^n$$
$$\psi^k := p_h(t_{n+1}) - p_{ht}^n \qquad e_p^n := p_h(t_n) - p_{ht}^n$$

and the increment operator

$$\delta_t a^n := a^n - a^{n-1}.$$

Now, we can proof a first result for the error between the error for the auxiliary velocity and its projection:

LEMMA 5.2: *For all $1 \leq m \leq N$ the difference between the velocity errors can be bounded as*

$$\|\boldsymbol{e}_u^m - \widetilde{\boldsymbol{e}}_u^m\|_0 \leq C(\Delta t)^2.$$

Proof: The error equation due to the diffusion step (5.6) reads

$$\begin{aligned}&\left(\frac{3\widetilde{\boldsymbol{e}}_u^n - 4\boldsymbol{e}_u^{n-1} + \boldsymbol{e}_u^{n-2}}{2\Delta t}, \boldsymbol{v}_h\right) + Ek(\nabla\widetilde{\boldsymbol{e}}_u^n, \nabla\boldsymbol{v}_h) + 2(\boldsymbol{\omega}^n \times \widetilde{\boldsymbol{e}}_u^n, \boldsymbol{v}_h)\\ &+ a_h(\boldsymbol{\omega}^n, \widetilde{\boldsymbol{e}}_u^n, \boldsymbol{\omega}^n, \boldsymbol{v}_h) + t_h(\widetilde{\boldsymbol{e}}_u^n, \boldsymbol{v}_h) = (\boldsymbol{R}^n, \boldsymbol{v}_h) - (\nabla\psi^{n-1}, \boldsymbol{v}_h).\end{aligned} \tag{5.19}$$

We note the identities

$$\begin{aligned}&2(\boldsymbol{\omega}^n \times \widetilde{\boldsymbol{e}}_u^n, \boldsymbol{v}_h) - 2(\boldsymbol{\omega}^{n-1} \times \widetilde{\boldsymbol{e}}_u^{n-1}, \boldsymbol{v}_h)\\ &= 2(\delta_t\boldsymbol{\omega}^n \times \widetilde{\boldsymbol{e}}_u^{n-1}, \boldsymbol{v}_h) + 2(\boldsymbol{\omega}^n \times \delta_t\widetilde{\boldsymbol{e}}_u^n, \boldsymbol{v}_h),\end{aligned} \tag{5.20}$$

$$\begin{aligned}&- a_h(\boldsymbol{\omega}^n, \widetilde{\boldsymbol{u}}_h^n, \boldsymbol{\omega}^n, \boldsymbol{v}_h) + a_h(\boldsymbol{\omega}^{n-1}, \widetilde{\boldsymbol{u}}_h^{n-1}, \boldsymbol{\omega}^{n-1}, \boldsymbol{v}_h)\\ &= a_h(\boldsymbol{\omega}^n, \widetilde{\boldsymbol{e}}_u^n, \boldsymbol{\omega}^n, \boldsymbol{v}_h) - a_h(\boldsymbol{\omega}^{n-1}, \widetilde{\boldsymbol{e}}_u^{n-1}, \boldsymbol{\omega}^{n-1}, \boldsymbol{v}_h)\\ &\quad - a_h(\boldsymbol{\omega}^n, \boldsymbol{u}_h(t_n), \boldsymbol{\omega}^n, \boldsymbol{v}_h) + a_h(\boldsymbol{\omega}^{n-1}, \boldsymbol{u}_h(t_{n-1}), \boldsymbol{\omega}^{n-1}, \boldsymbol{v}_h)\end{aligned} \tag{5.21}$$

and

$$\begin{aligned}&a_h(\boldsymbol{\omega}^n, \widetilde{\boldsymbol{e}}_u^n, \boldsymbol{\omega}^n, \boldsymbol{v}_h) - a_h(\boldsymbol{\omega}^{n-1}, \widetilde{\boldsymbol{e}}_u^{n-1}, \boldsymbol{\omega}^{n-1}, \boldsymbol{v}_h)\\ &= a_h(\boldsymbol{\omega}^n, \delta_t\widetilde{\boldsymbol{e}}_u^n, \boldsymbol{\omega}^n, \boldsymbol{v}_h) + a_h(\boldsymbol{\omega}^n, \widetilde{\boldsymbol{e}}_u^{n-1}, \boldsymbol{\omega}^n, \boldsymbol{v}_h)\\ &\quad - a_h(\boldsymbol{\omega}^{n-1}, \widetilde{\boldsymbol{e}}_u^{n-1}, \boldsymbol{\omega}^{n-1}, \boldsymbol{v}_h)\\ &= a_h(\boldsymbol{\omega}^n, \delta_t\widetilde{\boldsymbol{e}}_u^n, \boldsymbol{\omega}^n, \boldsymbol{v}_h) + a_h(\delta_t\boldsymbol{\omega}^n, \widetilde{\boldsymbol{e}}_u^{n-1}, \boldsymbol{\omega}^n, \boldsymbol{v}_h)\\ &\quad + a_h(\boldsymbol{\omega}^{n-1}, \widetilde{\boldsymbol{e}}_u^{n-1}, \boldsymbol{\omega}^n, \boldsymbol{v}_h) - a_h(\boldsymbol{\omega}^{n-1}, \widetilde{\boldsymbol{e}}_u^{n-1}, \boldsymbol{\omega}^{n-1}, \boldsymbol{v}_h)\\ &= a_h(\boldsymbol{\omega}^n, \delta_t\widetilde{\boldsymbol{e}}_u^n, \boldsymbol{\omega}^n, \boldsymbol{v}_h) + a_h(\delta_t\boldsymbol{\omega}^n, \widetilde{\boldsymbol{e}}_u^{n-1}, \boldsymbol{\omega}^n, \boldsymbol{v}_h)\\ &\quad + a_h(\boldsymbol{\omega}^{n-1}, \widetilde{\boldsymbol{e}}_u^{n-1}, \delta_t\boldsymbol{\omega}^n, \boldsymbol{v}_h)\end{aligned} \tag{5.22}$$

and get therefore

$$\begin{aligned}
&- a_h(\boldsymbol{\omega}^n, \widetilde{\boldsymbol{u}}_{ht}^n, \boldsymbol{\omega}^n, \boldsymbol{v}_h) + a_h(\boldsymbol{\omega}^{n-1}, \widetilde{\boldsymbol{u}}_{ht}^{n-1}, \boldsymbol{\omega}^{n-1}, \boldsymbol{v}_h) \\
&= a_h(\boldsymbol{\omega}^n, \delta_t \widetilde{\boldsymbol{e}}_u^n, \boldsymbol{\omega}^n, \boldsymbol{v}_h) + a_h(\delta_t \boldsymbol{\omega}^n, \widetilde{\boldsymbol{e}}_u^{n-1}, \boldsymbol{\omega}^n, \boldsymbol{v}_h) \\
&\quad + a_h(\boldsymbol{\omega}^{n-1}, \widetilde{\boldsymbol{e}}_u^{n-1}, \delta_t \boldsymbol{\omega}^n, \boldsymbol{v}_h) \\
&\quad - a_h(\boldsymbol{\omega}^n, \delta_t \boldsymbol{u}_h(t_n), \boldsymbol{\omega}^n, \boldsymbol{v}_h) - a_h(\delta_t \boldsymbol{\omega}^n, \boldsymbol{u}_h(t_{n-1}), \boldsymbol{\omega}^n, \boldsymbol{v}_h) \\
&\quad - a_h(\boldsymbol{\omega}^{n-1}, \boldsymbol{u}_h(t_{n-1}), \delta_t \boldsymbol{\omega}^n, \boldsymbol{v}_h).
\end{aligned} \tag{5.23}$$

From the fact that the increment operator is linear we get

$$\begin{aligned}
&\left(\frac{3\delta_t \widetilde{\boldsymbol{e}}_u^n - 4\delta_t \boldsymbol{e}_u^{n-1} + \delta_t \boldsymbol{e}_u^{n-2}}{2\Delta t}, \boldsymbol{v}_h \right) + Ek(\nabla \delta_t \widetilde{\boldsymbol{e}}_u^n, \nabla \boldsymbol{v}_h) \\
&\quad + 2(\boldsymbol{\omega}^n \times \delta_t \widetilde{\boldsymbol{e}}_u^n, \boldsymbol{v}_h) + a_h(\boldsymbol{\omega}^n, \delta_t \widetilde{\boldsymbol{e}}_u^n, \boldsymbol{\omega}^n, \boldsymbol{v}_h) \\
&\quad + \gamma(\nabla \cdot \delta_t \widetilde{\boldsymbol{e}}_u^n, \nabla \cdot \boldsymbol{v}_h) \\
&= (\delta_t \boldsymbol{R}^n, \boldsymbol{v}_h) - (\nabla \delta_t \psi^{n-1}, \boldsymbol{v}_h) + a_h(\boldsymbol{\omega}^n, \delta_t \boldsymbol{u}_h(t_n), \boldsymbol{\omega}^n, \boldsymbol{v}_h) \\
&\quad + a_h(\delta_t \boldsymbol{\omega}^n, \boldsymbol{u}_h(t_{n-1}), \boldsymbol{\omega}^n, \boldsymbol{v}_h) + a_h(\boldsymbol{\omega}^{n-1}, \boldsymbol{u}_h(t_{n-1}), \delta_t \boldsymbol{\omega}^n, \boldsymbol{v}_h) \\
&\quad + 2(\delta_t \boldsymbol{\omega}^n \times \widetilde{\boldsymbol{e}}_u^{n-1}, \boldsymbol{v}_h) + a_h(\delta_t \boldsymbol{\omega}^n, \widetilde{\boldsymbol{e}}_u^{n-1}, \boldsymbol{\omega}^n, \boldsymbol{v}_h) + a_h(\boldsymbol{\omega}^{n-1}, \widetilde{\boldsymbol{e}}_u^{n-1}, \delta_t \boldsymbol{\omega}^n, \boldsymbol{v}_h).
\end{aligned} \tag{5.24}$$

Now, we can do the same for the error in the projection step (5.7) and get

$$\left(\frac{3\delta_t \boldsymbol{e}_u^n - 3\delta_t \widetilde{\boldsymbol{e}}_u^n}{2\Delta t} + \nabla \delta_t e_p^n - \nabla \delta_t \psi^{n-1} - Ek \nabla \pi_p \nabla \cdot \delta_t \widetilde{\boldsymbol{u}}_{ht}^n, \nabla q_h \right) = 0. \tag{5.25}$$

Next we test the incremental error equation (5.24) with $4\Delta t \delta_t \widetilde{\boldsymbol{e}}_u^n$ to arrive at

$$\begin{aligned}
&(2(3\delta_t \widetilde{\boldsymbol{e}}_u^n - 4\delta_t \boldsymbol{e}_u^{n-1} + \delta_t \boldsymbol{e}_u^{n-2}), \delta_t \widetilde{\boldsymbol{e}}_u^n) + 4\Delta t Ek \|\nabla \delta_t \widetilde{\boldsymbol{e}}_u^n\|_0^2 \\
&\quad + 4\Delta t \gamma \|\nabla \cdot \delta_t \widetilde{\boldsymbol{e}}_u^n\|_0^2 + 4\Delta t \sum_M \alpha_M \|\kappa_M(\boldsymbol{\omega}^n \times \delta_t \widetilde{\boldsymbol{e}}_u^n)\|_0^2 \\
&= 4\Delta t((\delta_t p_{ht}^n - \nabla \delta_t \psi^{n-1}, \delta_t \widetilde{\boldsymbol{e}}_u^n) + a_h(\boldsymbol{\omega}^n, \delta_t \boldsymbol{u}_h(t_n), \boldsymbol{\omega}^n, \delta_t \widetilde{\boldsymbol{e}}_u^n) \\
&\quad + a_h(\delta_t \boldsymbol{\omega}^n, \boldsymbol{u}_h(t_{n-1}), \boldsymbol{\omega}^n, \delta_t \widetilde{\boldsymbol{e}}_u^n) + a_h(\boldsymbol{\omega}^{n-1}, \boldsymbol{u}_h(t_{n-1}), \delta_t \boldsymbol{\omega}^n, \delta_t \widetilde{\boldsymbol{e}}_u^n) \\
&\quad + 2(\delta_t \boldsymbol{\omega}^n \times \widetilde{\boldsymbol{e}}_u^{n-1}, \delta_t \widetilde{\boldsymbol{e}}_u^n) + a_h(\delta_t \boldsymbol{\omega}^n, \widetilde{\boldsymbol{e}}_u^{n-1}, \boldsymbol{\omega}^n, \delta_t \widetilde{\boldsymbol{e}}_u^n) \\
&\quad + a_h(\boldsymbol{\omega}^{n-1}, \widetilde{\boldsymbol{e}}_u^{n-1}, \delta_t \boldsymbol{\omega}^n, \delta_t \widetilde{\boldsymbol{e}}_u^n)) \\
&= 4\Delta t((\delta_t p_{ht}^n - \nabla \delta_t \psi^{n-1}, \delta_t \widetilde{\boldsymbol{e}}_u^n) + a_h(\boldsymbol{\omega}^n, \delta_t \boldsymbol{u}_h(t_n), \boldsymbol{\omega}^n, \delta_t \widetilde{\boldsymbol{e}}_u^n))
\end{aligned} \tag{5.26}$$

assuming that $\boldsymbol{\omega}$ is constant with respect to time. The first term is then splitted according to

$$\begin{aligned}
&(2(3\delta_t \widetilde{\boldsymbol{e}}_u^n - 4\delta_t \boldsymbol{e}_u^{n-1} + \delta_t \boldsymbol{e}_u^{n-2}), \delta_t \widetilde{\boldsymbol{e}}_u^n) = I_1 + I_2 + I_3 \\
&I_1 := 3\|\delta_t \widetilde{\boldsymbol{e}}_u^n\|_0^2 + 3\|\delta_t \boldsymbol{e}_u^n - \delta_t \widetilde{\boldsymbol{e}}_u^n\|_0^2 - 3\|\delta_t \boldsymbol{e}_u^n\|_0^2 \\
&I_2 := 2(\delta t \widetilde{\boldsymbol{e}}_u^n - \delta t \boldsymbol{e}_u^n, 3\delta_t \boldsymbol{e}_u^n - 4\delta_t \boldsymbol{e}_u^{n-1} + \delta_t \boldsymbol{e}_u^{n-2}) \\
&I_3 := \|\delta_t \boldsymbol{e}_u^n\|_0^2 + \|2\delta_t \boldsymbol{e}_u^n - \delta_t \boldsymbol{e}_u^{n-1}\|_0^2 + \|\delta_{ttt} \boldsymbol{e}_u^n\|_0^2 \\
&\qquad - \|\delta_t \boldsymbol{e}^{n-1}\|_0^2 - \|2\delta_t \boldsymbol{e}_u^{n-1} - \delta_t \boldsymbol{e}_u^{n-2}\|_0^2.
\end{aligned}$$

The terms I_1 and I_3 are treated exactly as in [2] and also the second term vanishes

$$\begin{aligned}\frac{3}{4\Delta t}I_2 &= (\nabla(\delta_t\psi^{n-1} - \delta_t e_p^n + Ek\nabla\pi_p\nabla\cdot\delta_t\widetilde{\boldsymbol{u}}_{ht}^n), 3\delta_t\boldsymbol{e}_u^n - 4\delta_t\boldsymbol{e}_u^{n-1} + \delta_t\boldsymbol{e}_u^{n-2}) \\ &= -(\delta_t\psi^{n-1} - \delta_t e_p^n + Ek\pi_p\nabla\cdot\delta_t\widetilde{\boldsymbol{u}}_{ht}^n), \nabla\cdot(3\delta_t\boldsymbol{e}_u^n - 4\delta_t\boldsymbol{e}_u^{n-1} + \delta_t\boldsymbol{e}_u^{n-2})) = 0\end{aligned}$$

due to the fact that $\boldsymbol{u}_h$ and $\boldsymbol{u}_{ht}$ are weakly divergence-free. The incremental projection error equation (5.25) may be rewritten as

$$\begin{aligned}&\left(\frac{3\delta_t\boldsymbol{e}_u^n}{2\Delta t} + \nabla\delta_t e_p^n - Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^n, \nabla q_h\right) \\ &= \left(\frac{3\delta_t\widetilde{\boldsymbol{e}}_u^n}{2\Delta t} + \nabla\delta_t\psi^{n-1} - Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^{n-1}, \nabla q_h\right).\end{aligned} \tag{5.27}$$

Testing both sides with themselves and multiplying with $\frac{4(\Delta t)^2}{3}$ yields

$$\begin{aligned}&\|3\delta_t\boldsymbol{e}_u^n\|_0^2 + \frac{4(\Delta t)^2}{3}\|\nabla\delta_t e_p^n - Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 \\ &= \|3\delta_t\widetilde{\boldsymbol{e}}_u^n\|_0^2 + \frac{4(\Delta t)^2}{3}\|\nabla\delta_t\psi^{n-1} - Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^{n-1}\|_0^2 \\ &\quad + 4\Delta t(\delta_t\widetilde{\boldsymbol{e}}_u^n, \nabla\delta_t\psi^{n-1} - Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^{n-1}).\end{aligned}$$

For the mixed term we again use that $\boldsymbol{u}_h$ is solenoidal

$$\begin{aligned}&-4\Delta t(\delta_t\widetilde{\boldsymbol{e}}_u^n, Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^{n-1}) = -4\Delta t(\pi_p\nabla\cdot\delta_t\widetilde{\boldsymbol{e}}_u^n, Ek\nabla\cdot(\widetilde{\boldsymbol{u}}_{ht}^{n-1} - \boldsymbol{u}_h(t_{n-1}))) \\ &= 4\Delta t(\delta_t\widetilde{\boldsymbol{e}}_u^n, Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{e}}_u^{n-1}) \\ &= 2Ek\Delta t(\|\pi_p\nabla\cdot\widetilde{\boldsymbol{e}}_u^{n-1}\|_0^2 - \|\pi_p\nabla\cdot\widetilde{\boldsymbol{e}}_u^n\|_0^2 + \|\pi_p\nabla\cdot\delta_t\widetilde{\boldsymbol{e}}_u^n\|_0^2) \\ &\leq 2Ek\Delta t(\|\pi_p\nabla\cdot\widetilde{\boldsymbol{e}}_u^{n-1}\|_0^2 - \|\pi_p\nabla\cdot\widetilde{\boldsymbol{e}}_u^n\|_0^2 + \|\nabla\delta_t\widetilde{\boldsymbol{e}}_u^n\|_0^2).\end{aligned}$$

Similarly we get

$$\begin{aligned}&\|\nabla\delta_t\psi^{n-1} - Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^{n-1}\|_0^2 \\ &= \|\nabla\delta_t e_p^{n-1} - Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^{n-1} + \nabla p_{tt}(t_n)\|_0^2 \\ &\leq (c(\Delta t)^2 + \|\nabla\delta_t e_p^{n-1} - Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^{n-1}\|_0)^2 \\ &\leq c(\Delta t)^4 + 2c(\Delta t)^2\|\nabla\delta_t e_p^{n-1} - Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^{n-1}\|_0 \\ &\quad + \|\nabla\delta_t e_p^{n-1} - Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^{n-1}\|_0)^2 \\ &\leq c(\Delta t)^4 + c(\Delta t)((\Delta t)^2 + \|\nabla\delta_t e_p^{n-1} - Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^{n-1}\|_0^2) \\ &\quad + \|\nabla\delta_t e_p^{n-1} - Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^{n-1}\|_0)^2 \\ &\leq c(\Delta t)^3 + (1 + c\Delta t)\|\nabla\delta_t e_p^{n-1} - Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^{n-1}\|_0^2.\end{aligned}$$

Combining all the estimates gives

$$
\begin{aligned}
&3\|\delta_t\widetilde{\boldsymbol{e}}_u^n\|_0^2 + 3\|\delta_t\boldsymbol{e}_u^n - \delta_t\widetilde{\boldsymbol{e}}_u^n\|_0^2 - 3\|\delta_t\boldsymbol{e}_u^n\|_0^2\\
&+ \|\delta_t\boldsymbol{e}_u^n\|_0^2 + \|2\delta_t\boldsymbol{e}_u^n - \delta_t\boldsymbol{e}_u^{n-1}\|_0^2 + \|\delta_{ttt}\boldsymbol{e}_u^n\|_0^2 - \|\delta_t\boldsymbol{e}^{n-1}\|_0^2 - \|2\delta_t\boldsymbol{e}_u^{n-1} - \delta_t\boldsymbol{e}_u^{n-2}\|_0^2\\
&+ 4\Delta t Ek\|\nabla\delta_t\widetilde{\boldsymbol{e}}_u^n\|_0^2 + 4\Delta t\gamma\|\nabla\cdot\delta_t\widetilde{\boldsymbol{e}}_u^n\|_0^2 + 4\Delta t\alpha\|\boldsymbol{\omega}_M\times\delta_t\widetilde{\boldsymbol{e}}_u^n\|_0^2\\
&+ \|3\delta_t\boldsymbol{e}_u^n\|_0^2 + \frac{4(\Delta t)^2}{3}\|\nabla\delta_t e_p^n - Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^n\|_0^2\\
&\le 4\Delta t(\delta_t\boldsymbol{R}^n, \delta_t\widetilde{\boldsymbol{e}}_u^n) + \|3\delta_t\widetilde{\boldsymbol{e}}_u^n\|_0^2 + 4\Delta t a_h(\boldsymbol{\omega}^n, \delta_t\boldsymbol{u}_h(t_n), \boldsymbol{\omega}^n, \delta_t\widetilde{\boldsymbol{e}}_u^n)\\
&\quad + \frac{4(\Delta t)^2}{3}(c(\Delta t)^3 + (1+c\Delta t)\|\nabla\delta_t e_p^{n-1} - Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^{n-1}\|_0^2)\\
&\quad + 2Ek\Delta t(\|\pi_p\nabla\cdot\widetilde{\boldsymbol{e}}_u^{n-1}\|_0^2 - \|\pi_p\nabla\cdot\widetilde{\boldsymbol{e}}_u^n\|_0^2 + \|\nabla\delta_t\widetilde{\boldsymbol{e}}_u^n\|_0^2) + 4\Delta t(\delta_t\widetilde{\boldsymbol{e}}_u^n, \nabla\delta_t\psi^{n-1})
\end{aligned}
$$

and using $\Delta t\|\delta_t\widetilde{\boldsymbol{e}}_u^n\|_0^2 \le 2\Delta t\|\delta_t\boldsymbol{e}_u^n\|_0^2 + 2\|\delta_t\boldsymbol{e}_u^n - \delta_t\widetilde{\boldsymbol{e}}_u^n\|_0^2$ for $\Delta t < 1$ we get

$$
\begin{aligned}
&\|\delta_t\boldsymbol{e}_u^n - \delta_t\widetilde{\boldsymbol{e}}_u^n\|_0^2 + \|\delta_t\boldsymbol{e}_u^n\|_0^2 + \|2\delta_t\boldsymbol{e}_u^n - \delta_t\boldsymbol{e}_u^{n-1}\|_0^2 + \|\delta_{ttt}\boldsymbol{e}_u^n\|_0^2 + 2\Delta t Ek\|\nabla\delta_t\widetilde{\boldsymbol{e}}_u^n\|_0^2\\
&+ 4\Delta t\gamma\|\nabla\cdot\delta_t\widetilde{\boldsymbol{e}}_u^n\|_0^2 + 2\Delta t\sum_M\big(\alpha_M\|\kappa_M(\boldsymbol{\omega}_M\times\delta_t\widetilde{\boldsymbol{e}}_u^n)\|_{0,M}^2\big) + 2Ek\Delta t\|\pi_p\nabla\cdot\widetilde{\boldsymbol{e}}_u^n\|_0^2\\
&+ \frac{4(\Delta t)^2}{3}\|\nabla\delta_t e_p^n - Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^n\|_0^2\\
&\le 2\Delta t\|\delta_t\boldsymbol{R}^n\|_0^2 + 2\Delta t\|\delta_t\boldsymbol{e}_u^n\|_0^2 + 2\Delta t\sum_M\big(\alpha_M\|\kappa_M(\boldsymbol{\omega}^n\times\delta_t\boldsymbol{u}_h(t_n))\|_{0,M}^2\big)\\
&\quad + \|\delta_t\boldsymbol{e}^{n-1}\|_0^2 + \|2\delta_t\boldsymbol{e}_u^{n-1} - \delta_t\boldsymbol{e}_u^{n-2}\|_0^2 + 2Ek\Delta t\|\pi_p\nabla\cdot\widetilde{\boldsymbol{e}}_u^{n-1}\|_0^2\\
&\quad + \frac{4(\Delta t)^2}{3}(c(\Delta t)^3 + (1+c\Delta t)\|\nabla\delta_t e_p^{n-1} - Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^{n-1}\|_0^2).
\end{aligned}
$$

Summing over n yields

$$
\begin{aligned}
&\|\delta_t \boldsymbol{e}_u^N\|_0^2 + \|2\delta_t \boldsymbol{e}_u^N - \delta_t \boldsymbol{e}_u^{N-1}\|_0^2 \\
&\quad + \frac{4(\Delta t)^2}{3}\|\nabla\delta_t e_p^N - Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^N\|_0^2 + 2Ek\Delta t\|\pi_p\nabla\cdot\widetilde{\boldsymbol{e}}_u^N\|_0^2 + \\
&\sum_{n=3}^{N}(\|\delta_t \boldsymbol{e}_u^n - \delta_t\widetilde{\boldsymbol{e}}_u^n\|_0^2 + \|\delta_{ttt}\boldsymbol{e}_u^n\|_0^2 + 2\Delta t Ek\|\nabla\delta_t\widetilde{\boldsymbol{e}}_u^n\|_0^2 + 4\Delta t\gamma\|\nabla\cdot\delta_t\widetilde{\boldsymbol{e}}_u^n\|_0^2 \\
&\quad + 4\Delta t\sum_M\left(\alpha_M\|\kappa_M(\boldsymbol{\omega}\times\delta_t\widetilde{\boldsymbol{e}}_u^n)\|_{0,M}^2\right)) \\
&\leq \|\delta_t\boldsymbol{e}_u^2\|_0^2 + \|2\delta_t\boldsymbol{e}_u^2 - \delta_t\boldsymbol{e}_u^1\|_0^2 \\
&\quad + \frac{4(\Delta t)^2}{3}\|\nabla\delta_t e_p^4 - Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^4\|_0^2 + 2Ek\Delta t\|\pi_p\nabla\cdot\widetilde{\boldsymbol{e}}_u^4\|_0^2 + \\
&\quad \sum_{n=3}^{N}(2\Delta t\|\delta_t\boldsymbol{R}^n\|_0^2 + 2\Delta t\|\delta_t\boldsymbol{e}_u^n\| \\
&\qquad + \frac{4(\Delta t)^2}{3}(c(\Delta t)^3 + c\Delta t\|\nabla\delta_t e_p^{n-1} - Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^{n-1}\|_0^2) \\
&\qquad + 2\Delta t\sum_M\left(\alpha_M\|\kappa_M(\boldsymbol{\omega}^n\times\delta_t\boldsymbol{u}_h(t_n))\|_{0,M}^2\right)).
\end{aligned}
$$

Using the discrete Gronwall lemma for $\|\delta_t\boldsymbol{e}_u^n\|_0^2 + \frac{4(\Delta t)^2}{3}\|\nabla\delta_t e_p^n - Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^n\|_0^2 + 2Ek\Delta t\|\pi_p\nabla\cdot\widetilde{\boldsymbol{e}}_u^n\|_0^2$ we arrive at

$$
\begin{aligned}
&\|\delta_t\boldsymbol{e}_u^N\|_0^2 + \|2\delta_t\boldsymbol{e}_u^N - \delta_t\boldsymbol{e}_u^{N-1}\|_0^2 + \frac{4(\Delta t)^2}{3}\|\nabla\delta_t e_p^N - Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^N\|_0^2 \\
&\quad + 2Ek\Delta t\|\pi_p\nabla\cdot\widetilde{\boldsymbol{e}}_u^N\|_0^2 \\
&\quad + \sum_{n=3}^{N}(\|\delta_t\boldsymbol{e}_u^n - \delta_t\widetilde{\boldsymbol{e}}_u^n\|_0^2 + \|\delta_{ttt}\boldsymbol{e}_u^n\|_0^2 + 2\Delta t Ek\|\nabla\delta_t\widetilde{\boldsymbol{e}}_u^n\|_0^2 \\
&\qquad + 4\Delta t\gamma\|\nabla\cdot\delta_t\widetilde{\boldsymbol{e}}_u^n\|_0^2 + 4\Delta t a_h(\boldsymbol{\omega}, \delta_t\widetilde{\boldsymbol{e}}_u^n, \boldsymbol{\omega}, \delta_t\widetilde{\boldsymbol{e}}_u^n)) \\
&\leq (\|\delta_t\boldsymbol{e}_u^2\|_0^2 + \|2\delta_t\boldsymbol{e}_u^2 - \delta_t\boldsymbol{e}_u^1\|_0^2 + \frac{4(\Delta t)^2}{3}\|\nabla\delta_t e_p^2 - Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^2\|_0^2 \\
&\quad + 2Ek\Delta t\|\pi_p\nabla\cdot\widetilde{\boldsymbol{e}}_u^2\|_0^2 \\
&\quad + \sum_{n=3}^{N}(2\Delta t\|\delta_t\boldsymbol{R}^n\|_0^2 + c(\Delta t)^5 \\
&\qquad + 2\Delta t\sum_M\left(\alpha_M\|\kappa_M(\boldsymbol{\omega}^n\times\delta_t\boldsymbol{u}_h(t_n))\|_{0,M}^2\right))) \exp(T/(1-\Delta t)) \\
&\leq C(\Delta t)^4 + C(\Delta t)^2 h^{2k+2} \leq C(\Delta t)^4
\end{aligned}
\tag{5.28}
$$

using the initial errors and

$$
2\Delta t\sum_M\left(\alpha_M\|\kappa_M(\boldsymbol{\omega}^n\times\delta_t\boldsymbol{u}_h(t_n))\|_{0,M}^2\right) \leq C(\Delta t)^3 h^{2k+2}.
$$

as well as $h^{k+1} \leq \Delta t$.

The error increment equation (5.25) states

$$\begin{aligned}\|\boldsymbol{e}_u^n - \widetilde{\boldsymbol{e}}_u^n\|_0 &= \frac{2\Delta t}{3}\|\nabla(\delta_t e_p^n - \delta_t p_h(t_{k+1}) - Ek\pi_p\nabla\cdot\delta_t\widetilde{\boldsymbol{u}}_{ht}^n)\|_0 \\ &\leq \frac{2\Delta t}{3}(\|\nabla(\delta_t e_p^n + Ek\pi_p\nabla\cdot\delta_t\widetilde{\boldsymbol{u}}_{ht}^n)\|_0 + \|\delta_t p_h(t_{k+1})\|_0) \\ &\leq C(\Delta t)^2.\end{aligned}$$

□

5.4. Velocity Error Estimate

Next we want to bound $\|\widetilde{\boldsymbol{e}}_u^n\|$. Therefore we eliminate $\boldsymbol{e}_u^n$ in (5.19) to give

$$\begin{aligned}&\left(\frac{3\widetilde{\boldsymbol{e}}_u^n - 4\widetilde{\boldsymbol{e}}_u^{n-1} + \widetilde{\boldsymbol{e}}_u^{n-2}}{2\Delta t}, \boldsymbol{v}_h\right) + Ek(\nabla\widetilde{\boldsymbol{e}}_u^n, \nabla\boldsymbol{v}_h) + 2(\boldsymbol{\omega}\times\widetilde{\boldsymbol{e}}_u^n, \boldsymbol{v}_h) \\ &+ a_h(\boldsymbol{\omega}, \widetilde{\boldsymbol{e}}_u^n, \boldsymbol{\omega}, \boldsymbol{v}_h) + \gamma(\nabla\cdot\widetilde{\boldsymbol{e}}_u^n, \nabla\cdot\boldsymbol{v}_h) - (\boldsymbol{R}^n, \boldsymbol{v}_h) \\ &= (\nabla(-p_h(t_n) + \frac{7}{3}p_{ht}^{n-1} - \frac{5}{3}p_{ht}^{n-2} + \frac{1}{3}p_{ht}^{n-3} \\ &\quad + \frac{4}{3}Ek\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^{n-1} - \frac{1}{3}Ek\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^{n-2}), \boldsymbol{v}_h) \\ &=: (\nabla\zeta^n, \boldsymbol{v}_h)\end{aligned} \tag{5.29}$$

This allows us to derive the desired estimate:

Lemma 5.3: *The total velocity converges according to*

$$\|\widetilde{\boldsymbol{e}}_u\|^2_{l^2(0,T;L^2(\Omega))} = \Delta t\sum_{n=0}^{N}\|\widetilde{\boldsymbol{e}}_u^n\|_0^2 \leq c(\Delta t)^4. \tag{5.30}$$

Proof: We test equation (5.29) with the inverse Stokes operator applied to $4\Delta t\widetilde{\boldsymbol{e}}_u^n$

$$\begin{aligned}&(2(3\widetilde{\boldsymbol{e}}_u^n - 4\widetilde{\boldsymbol{e}}_u^{n-1} + \widetilde{\boldsymbol{e}}_u^{n-2}), S\widetilde{\boldsymbol{e}}_u^n) + 4\Delta tEk(\nabla\widetilde{\boldsymbol{e}}_u^n, \nabla S\widetilde{\boldsymbol{e}}_u^n) + 8\Delta t(\boldsymbol{\omega}\times\widetilde{\boldsymbol{e}}_u^n, S\widetilde{\boldsymbol{e}}_u^n) \\ &\quad + 4\Delta t a_h(\boldsymbol{\omega}, \widetilde{\boldsymbol{e}}_u^n, \boldsymbol{\omega}, S\widetilde{\boldsymbol{e}}_u^n) + 4\Delta t\gamma(\nabla\cdot\widetilde{\boldsymbol{e}}_u^n, \nabla\cdot S\widetilde{\boldsymbol{e}}_u^n) \\ &= 4\Delta t(\boldsymbol{R}^n, S\widetilde{\boldsymbol{e}}_u^n) + 4\Delta t(\nabla\zeta^n, S\widetilde{\boldsymbol{e}}_u^n) = 4\Delta t(\boldsymbol{R}^n, S\widetilde{\boldsymbol{e}}_u^n).\end{aligned}$$

due to $S\widetilde{\boldsymbol{e}}_u^n \in \boldsymbol{V}_h^{div}$. For the discrete time derivative we notice that from the splitting we used before only I_3 remains.

$$\begin{aligned}(2(3\widetilde{\boldsymbol{e}}_u^n - 4\widetilde{\boldsymbol{e}}_u^{n-1} + \widetilde{\boldsymbol{e}}_u^{n-2}), S\widetilde{\boldsymbol{e}}_u^n) = \|\boldsymbol{e}_u^n\|_*^2 + \|2\boldsymbol{e}_u^n - \boldsymbol{e}_u^{n-1}\|_*^2 + \|\delta_{tt}\boldsymbol{e}_u^n\|_*^2 \\ - \|\boldsymbol{e}^{n-1}\|_*^2 - \|2\boldsymbol{e}_u^{n-1} - \boldsymbol{e}_u^{n-2}\|_*^2.\end{aligned}$$

Using the definition of the semi-norm induced by the inverse Stokes operator we get

$$\begin{aligned}
&|\widetilde{\boldsymbol{e}}_u^n|_*^2 + |2\widetilde{\boldsymbol{e}}_u^n - \widetilde{\boldsymbol{e}}_u^{n-1}|_*^2 + |\delta_{tt}\widetilde{\boldsymbol{e}}_u^n|_*^2 + 4\Delta t Ek(\nabla\widetilde{\boldsymbol{e}}_u^n, \nabla S\widetilde{\boldsymbol{e}}_u^n)\\
&+ 8\Delta t(\boldsymbol{\omega}\times\widetilde{\boldsymbol{e}}_u^n, S\widetilde{\boldsymbol{e}}_u^n) + 4\Delta t a_h(\boldsymbol{\omega}, \widetilde{\boldsymbol{e}}_u^n, \boldsymbol{\omega}, S\widetilde{\boldsymbol{e}}_u^n) + 4\Delta t\gamma(\nabla\cdot\widetilde{\boldsymbol{e}}_u^n, \nabla\cdot S\widetilde{\boldsymbol{e}}_u^n)\\
&= 4\Delta t(\boldsymbol{R}^n, S\widetilde{\boldsymbol{e}}_u^n) + |\widetilde{\boldsymbol{e}}_u^{n-1}|_*^2 + |2\widetilde{\boldsymbol{e}}_u^{n-1} - \widetilde{\boldsymbol{e}}_u^{n-2}|_*^2.
\end{aligned} \tag{5.31}$$

The consistency error can be bounded as

$$\begin{aligned}
4\Delta t(\boldsymbol{R}^n, S\widetilde{\boldsymbol{e}}_u^n) &\le 4\frac{\Delta t}{Ek}\|\boldsymbol{R}^n\|_{-1}^2 + \Delta t Ek\|S\widetilde{\boldsymbol{e}}_u^n\|_1^2\\
&\overset{(A.1)}{\le} 4\frac{\Delta t}{Ek}\|\boldsymbol{R}^n\|_{-1}^2 + \Delta t\|\widetilde{\boldsymbol{e}}_u^n\|_0^2.
\end{aligned}$$

Using (A.4) with $\epsilon = 2\left(2 + \frac{\gamma}{Ek} + \left(\frac{\max_M\{\sqrt{\alpha_M}\}}{\sqrt{Ek}} + \frac{2}{Ek}\right)\|\boldsymbol{\omega}_M\|\right)^{-2}$, the diffusive term, the grad-div stabilization, the Coriolis term and the Coriolis stabilization can be estimated by

$$\begin{aligned}
&4\Delta t Ek(\nabla\widetilde{\boldsymbol{e}}_u^n, \nabla S\widetilde{\boldsymbol{e}}_u^n) + 4\Delta t\gamma(\nabla\cdot\widetilde{\boldsymbol{e}}_u^n, \nabla\cdot S\widetilde{\boldsymbol{e}}_u^n)\\
&\quad + 4\Delta t a_h(\boldsymbol{\omega}, \widetilde{\boldsymbol{e}}_u^n, \boldsymbol{\omega}, S\widetilde{\boldsymbol{e}}_u^n) + 8\Delta t(\boldsymbol{\omega}\times S\widetilde{\boldsymbol{e}}_u^n)\\
&\ge 2\Delta t\|\widetilde{\boldsymbol{e}}_u^n\|_0^2 - c\Delta t\|\widetilde{\boldsymbol{e}}_u^n - \boldsymbol{e}_u^n\|_0^2.
\end{aligned}$$

where $c = 2\left(2 + \frac{\gamma}{Ek} + \left(\frac{\max_M\{\sqrt{\alpha_M}\}}{\sqrt{Ek}} + \frac{2}{Ek}\right)\|\boldsymbol{\omega}_M\|\right)^2$. Combining these estimates we arrive at

$$\begin{aligned}
&|\widetilde{\boldsymbol{e}}_u^n|_*^2 + |2\widetilde{\boldsymbol{e}}_u^n - \widetilde{\boldsymbol{e}}_u^{n-1}|_*^2 + |\delta_{tt}\widetilde{\boldsymbol{e}}_u^n|_*^2 + \Delta t\|\widetilde{\boldsymbol{e}}_u^n\|_0^2\\
&\le 4\frac{\Delta t}{Ek}\|\boldsymbol{R}^n\|_{-1}^2 + |\widetilde{\boldsymbol{e}}_u^{n-1}|_*^2 + |2\widetilde{\boldsymbol{e}}_u^{n-1} - \widetilde{\boldsymbol{e}}_u^{n-2}|_*^2 + c\Delta t\|\widetilde{\boldsymbol{e}}_u^n - \boldsymbol{e}_u^n\|_0^2
\end{aligned}$$

that yields summed up

$$\begin{aligned}
&|\widetilde{\boldsymbol{e}}_u^N|_*^2 + |2\widetilde{\boldsymbol{e}}_u^N - \widetilde{\boldsymbol{e}}_u^{N-1}|_*^2 + \sum_{n=3}^{N}(|\delta_{tt}\widetilde{\boldsymbol{e}}_u^n|_*^2 + \Delta t\|\widetilde{\boldsymbol{e}}_u^n\|_0^2)\\
&\le |\widetilde{\boldsymbol{e}}_u^2|_*^2 + |2\widetilde{\boldsymbol{e}}_u^2 - \widetilde{\boldsymbol{e}}_u^1|_*^2 + \sum_{n=3}^{N}(4\frac{\Delta t}{Ek}\|\boldsymbol{R}^n\|_{-1}^2 + c\Delta t\|\widetilde{\boldsymbol{e}}_u^n - \boldsymbol{e}_u^n\|_0^2) \le c(\Delta t)^4.
\end{aligned}$$

In particular we derive

$$\|\widetilde{\boldsymbol{e}}_u\|_{l^2(0,T;L^2(\Omega))}^2 = \Delta t\sum_{n=0}^{N}\|\widetilde{\boldsymbol{e}}_u^n\|_0^2 \le c(\Delta t)^4. \tag{5.32}$$

□

5.5. Error Estimate for the Discrete Time Derivative

In the previous estimates we could have derived that the velocity in the LPS converges linearly. In order improve this result it is important to get proper bound on the discrete time derivative.

LEMMA 5.4: *For all $1 \leq m \leq N$ the error of the discrete time derivative can be bounded according to*

$$\|D_t \widetilde{\boldsymbol{e}}_u\|^2_{l^2(0,T;L^2(\Omega))} \leq C(\Delta t)^5. \tag{5.33}$$

Proof: We start again from the error equation (5.31) for the diffusive step in which the error $\boldsymbol{u}^n_{ht}$ is eliminated. This time we apply the increment operator and test with $4\Delta t \delta_t \widetilde{\boldsymbol{e}}^n_u$. This gives

$$\begin{aligned}
&|\delta_t \widetilde{\boldsymbol{e}}^n_u|^2_* + |2\delta_t \widetilde{\boldsymbol{e}}^n_u - 2\delta_t \widetilde{\boldsymbol{e}}^{n-1}_u|^2_* + |\delta_{ttt} \widetilde{\boldsymbol{e}}^n_u|^2_* + 4\Delta t Ek(\nabla \delta_t \widetilde{\boldsymbol{e}}^n_u, \nabla S \delta_t \widetilde{\boldsymbol{e}}^n_u) \\
&\quad + 8\Delta t(\boldsymbol{\omega} \times \widetilde{\boldsymbol{e}}^n_u, \delta_t S \widetilde{\boldsymbol{e}}^n_u) + 4\Delta t a_h(\boldsymbol{\omega}, \widetilde{\boldsymbol{e}}^n_u, \boldsymbol{\omega}, \delta_t S \widetilde{\boldsymbol{e}}^n_u) \\
&\quad + 4\Delta t \gamma(\nabla \cdot \delta_t \widetilde{\boldsymbol{e}}^n_u, \nabla \cdot S \delta_t \widetilde{\boldsymbol{e}}^n_u) \\
&= 4\Delta t(\delta_t \boldsymbol{R}^n, S\delta_t \widetilde{\boldsymbol{e}}^n_u) + |\delta_t \widetilde{\boldsymbol{e}}^{n-1}_u|^2_* + |2\delta_t \widetilde{\boldsymbol{e}}^{n-1}_u - 2\delta_t \widetilde{\boldsymbol{e}}^{n-2}_u|^2_*
\end{aligned}$$

assuming $\boldsymbol{\omega}$ does not depend on time.
Using the same tricks as above we arrive at

$$\begin{aligned}
&|\delta_t \widetilde{\boldsymbol{e}}^n_u|^2_* + |2\delta_t \widetilde{\boldsymbol{e}}^n_u - 2\delta_t \widetilde{\boldsymbol{e}}^{n-1}_u|^2_* + |\delta_{ttt} \widetilde{\boldsymbol{e}}^n_u|^2_* + \Delta t \|\delta_t \widetilde{\boldsymbol{e}}^n_u\|^2_0 \\
&\leq 4\Delta t \|\delta_t \boldsymbol{R}^n\|^2_{-1} + |\delta_t \widetilde{\boldsymbol{e}}^{n-1}_u|^2_* + |2\delta_t \widetilde{\boldsymbol{e}}^{n-1}_u - 2\delta_t \widetilde{\boldsymbol{e}}^{n-2}_u|^2_* + c\Delta t \|\delta_t \widetilde{\boldsymbol{e}}^n_u - \delta_t \boldsymbol{e}^n_u\|^2_0
\end{aligned}$$

that yields summed up

$$\begin{aligned}
&|\delta_t \widetilde{\boldsymbol{e}}^N_u|^2_* + |2\delta_t \widetilde{\boldsymbol{e}}^N_u - 2\delta_t \widetilde{\boldsymbol{e}}^{N-1}_u|^2_* + \sum_{n=3}^N (|\delta_{ttt} \widetilde{\boldsymbol{e}}^n_u|^2_* + \Delta t \|\delta_t \widetilde{\boldsymbol{e}}^n_u\|^2_0) \\
&\leq |\delta_t \widetilde{\boldsymbol{e}}^2_u|^2_* + |2\delta_t \widetilde{\boldsymbol{e}}^1_u - 2\delta_t \widetilde{\boldsymbol{e}}^2_u|^2_* + \sum_{n=3}^N (4\frac{\Delta t}{Ek} \|\delta_t \boldsymbol{R}^n\|^2_{-1} + c\Delta t \|\delta_t \widetilde{\boldsymbol{e}}^n_u - \delta_t \boldsymbol{e}^n_u\|^2_0) \\
&\leq c(\Delta t)^5.
\end{aligned}$$

In particular we derive

$$\|\delta_t \widetilde{\boldsymbol{e}}_u\|^2_{l^2(0,T;L^2(\Omega))} = \Delta t \sum_{n=0}^N \|\delta_t \widetilde{\boldsymbol{e}}^n_u\|^2_0 \leq c(\Delta t)^5 \tag{5.34}$$

and due to $D_t \widetilde{\boldsymbol{e}}^n_u = \frac{3}{2}\delta_t \widetilde{\boldsymbol{e}}^n_u - \frac{1}{2}\delta_t \widetilde{\boldsymbol{e}}^{n-1}_u$ the discrete time derivative can be bounded according to

$$\|D_t \widetilde{\boldsymbol{e}}_u\|^2_{l^2(0,T;L^2(\Omega))} \leq \Delta t \sum_{n=0}^N (3\|\delta_t \widetilde{\boldsymbol{e}}^n_u\|^2_0 + \|\delta_t \widetilde{\boldsymbol{e}}^{n-1}_u\|^2_0) \leq c(\Delta t)^5. \tag{5.35}$$

□

5.6. Final Estimates for Velocity and Pressure

Now, we proved almost everything we wanted. The only things that are left are the estimated in the H^1-Seminorm for the velocity and the pressure estimates. These of course rely heavily on the inf-sup stability (2.3) of the used ansatz spaces. In particular this is equivalent to the surjectivity of the discrete divergence operator.

Lemma 5.5: *For all $1 \leq m \leq N$ the velocity error in the LPS-norm and the pressure error in the $L^2(\Omega)$-norm converge according to*

$$\|\boldsymbol{e}_u^m\|_{LPS}^2 + \|e_p^m\|_0^2 \leq C(\Delta t)^3. \tag{5.36}$$

Proof: The error equations for the diffusive and for the projective step are equivalent to the inhomogeneous Stokes problem

$$\begin{aligned}
&(\nabla e_p^n - Ek\nabla\pi_p\nabla\cdot\widetilde{\boldsymbol{e}}_u^n, \boldsymbol{v}_h) + Ek(\nabla\widetilde{\boldsymbol{e}}_u^n, \nabla\boldsymbol{v}_h) + 2(\boldsymbol{\omega}\times\widetilde{\boldsymbol{e}}_u^n, \boldsymbol{v}_h)\\
&+ a_h(\boldsymbol{\omega}, \widetilde{\boldsymbol{e}}_u^n, \boldsymbol{\omega}, \boldsymbol{v}_h) + \gamma(\nabla\cdot\widetilde{\boldsymbol{e}}_u^n, \nabla\cdot\boldsymbol{v}_h)\\
&= (\boldsymbol{R}^n, \boldsymbol{v}_h) - \left(\frac{3\boldsymbol{e}_u^n - 4\boldsymbol{e}_u^{n-1} + \boldsymbol{e}_u^{n-2}}{2\Delta t}, \boldsymbol{v}_h\right), \boldsymbol{v}_h) =: (h^n, \boldsymbol{v}_h)\\
(\nabla\cdot\widetilde{\boldsymbol{e}}_u^n, q_h) &= (\pi_p\nabla\cdot\widetilde{\boldsymbol{e}}_u^n, q_h)\\
&= \frac{2\Delta t}{3}(\nabla(p_{ht}^n - p_{ht}^{n-1} + Ek\pi_p\nabla\cdot\widetilde{\boldsymbol{u}}_{ht}^n), \nabla q_h) =: (g^n, q_h)\\
\widetilde{\boldsymbol{e}}_u^n|_{\partial\Omega} &= 0.
\end{aligned} \tag{5.37}$$

Due to (5.28) we know

$$\|g^n\|_0^2 = \|\pi_p\nabla\cdot\widetilde{\boldsymbol{e}}_u^n\|_0^2 \leq C(\Delta t)^3 \min\left\{\frac{1}{Ek}, \frac{1}{\gamma\Delta t}\right\} \tag{5.38}$$

and noticing $\boldsymbol{e}_u^n = P_H(\widetilde{\boldsymbol{e}}_u^n)$

$$\begin{aligned}
\Delta t\sum_{n=0}^{N}\|h^n\|_{-1}^2 &\lesssim \Delta t\sum_{n=0}^{N}\left(\|\boldsymbol{R}^n\|_{-1}^2 + \frac{\|D_t\boldsymbol{e}_u^n\|_{-1}^2}{(\Delta t)^2}\right)\\
&\lesssim \Delta t\sum_{n=0}^{N}\left(\|\boldsymbol{R}^n\|_{-1}^2 + \frac{\|D_t\widetilde{\boldsymbol{e}}_u^n\|_{-1}^2}{(\Delta t)^2}\right) \overset{(5.35)}{\leq} C(\Delta t)^3.
\end{aligned} \tag{5.39}$$

Finally, we need a stability result for such a grad-div stabilized Stokes equation defined by

$$\begin{aligned}
Ek(\nabla\boldsymbol{u}_h, \nabla\boldsymbol{v}_h) - (p_h, \nabla\cdot\boldsymbol{v}_h) + 2(\boldsymbol{\omega}\times\boldsymbol{u}_h, \boldsymbol{v}_h)&\\
+a_h(\boldsymbol{\omega}, \boldsymbol{u}_h, \boldsymbol{\omega}, \boldsymbol{v}_h) + \gamma(\nabla\cdot\boldsymbol{u}_h, \nabla\cdot\boldsymbol{v}_h) &= (h, \boldsymbol{v}_h) \quad \forall\boldsymbol{v}_h\in\boldsymbol{V}_h\\
(\nabla\cdot\boldsymbol{u}_h, q_h) &= (g, q_h) \quad \forall q_h\in Q_h\\
\boldsymbol{u}_h|_{\partial\Omega} &= 0.
\end{aligned} \tag{5.40}$$

Due to the discrete inf-sup condition there exists $\boldsymbol{u}_0 \in \boldsymbol{V}_h$ satisfying

$$\|\nabla \boldsymbol{u}_0\| \leq \frac{\|g\|_0}{\beta} \qquad (\nabla\cdot\boldsymbol{u}_0, q_h) = (g, q_h) \quad \forall q_h \in Q_h. \tag{5.41}$$

This means that $\boldsymbol{w}_h = \boldsymbol{u}_h - \boldsymbol{u}_0$ satisfies

$$\begin{aligned}
&Ek(\nabla\boldsymbol{w}_h, \nabla\boldsymbol{v}_h) - (p_h, \nabla\cdot\boldsymbol{v}_h) + (\nabla\cdot\boldsymbol{w}_h, q_h) + 2(\boldsymbol{\omega}\times\boldsymbol{w}_h, \boldsymbol{v}_h)\\
&+ a_h(\boldsymbol{\omega}, \boldsymbol{w}_h, \boldsymbol{\omega}, \boldsymbol{v}_h) + \gamma(\nabla\cdot\boldsymbol{w}_h, \nabla\cdot\boldsymbol{v}_h)\\
&= (h, \boldsymbol{v}_h) - Ek(\nabla\boldsymbol{u}_0, \nabla\boldsymbol{v}_h) - \gamma(\nabla\cdot\boldsymbol{u}_0, \nabla\cdot\boldsymbol{v}_h) - 2(\boldsymbol{\omega}\times\boldsymbol{u}_0, \boldsymbol{v}_h)\\
&- a_h(\boldsymbol{\omega}, \boldsymbol{u}_0, \boldsymbol{\omega}, \boldsymbol{v}_h)
\end{aligned} \tag{5.42}$$

for all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times Q_h$.

Testing symmetrically we get

$$\begin{aligned}
&Ek\|\nabla\boldsymbol{w}_h\|_0^2 + \gamma\|\nabla\cdot\boldsymbol{w}_h\|_0^2 + \sum_M \left(\alpha_M\|\kappa_M(\boldsymbol{\omega}_M\times\boldsymbol{w}_h)\|_{0,M}^2\right)\\
&= Ek\|\nabla\boldsymbol{w}_h\|_0^2 - (p_h, \nabla\cdot\boldsymbol{w}_h) + 2(\boldsymbol{\omega}\times\boldsymbol{w}_h, \boldsymbol{w}_h) + (\nabla\cdot\boldsymbol{w}_h, p_h)\\
&\quad + \sum_M \left(\alpha_M\|\kappa_M(\boldsymbol{\omega}_M\times\boldsymbol{w}_h)\|_{0,M}^2\right) + \gamma\|\nabla\cdot\boldsymbol{w}_h\|_0^2\\
&= (h, \boldsymbol{w}_h) + Ek(\nabla\boldsymbol{u}_0, \nabla\boldsymbol{w}_h) + \gamma(\nabla\cdot\boldsymbol{u}_0, \nabla\cdot\boldsymbol{w}_h)\\
&\quad - 2(\boldsymbol{\omega}\times\boldsymbol{u}_0, \boldsymbol{w}_h) - a_h(\boldsymbol{\omega}, \boldsymbol{u}_0, \boldsymbol{\omega}, \boldsymbol{w}_h)\\
&\leq (\|h\|_{-1} + Ek\|\nabla\boldsymbol{u}_0\|_0 + \gamma\|\nabla\cdot\boldsymbol{u}_0\|_0\\
&\quad + (2 + \max_M\{\alpha_M|\boldsymbol{\omega}_M|\})\|\boldsymbol{\omega}\|_\infty\|\nabla\boldsymbol{u}_0\|_0)\|\nabla\boldsymbol{w}_h)\|_0\\
&\leq \left(\|h\|_{-1} + \frac{Ek + \gamma + (2+\max_M\{\alpha_M|\boldsymbol{\omega}_M|\})\|\boldsymbol{\omega}\|_\infty}{\beta}\|g\|_0\right)\|\nabla\boldsymbol{w}_h)\|_0\\
\Rightarrow \|\nabla\boldsymbol{w}_h\|_0 &\leq \left(\frac{\|h\|_{-1}}{Ek} + \frac{Ek + \gamma + (2+\max_M\{\alpha_M|\boldsymbol{\omega}_M|\})\|\boldsymbol{\omega}\|_\infty}{Ek\beta}\|g\|_0\right)\\
\Rightarrow \|\nabla\boldsymbol{u}_h\|_0 &\lesssim \|\nabla\boldsymbol{u}_0\|_0^2 + \|\nabla\boldsymbol{w}_h\|_0^2\\
&\lesssim \left(\frac{\|h\|_{-1}}{Ek} + \frac{Ek + \gamma + (2+\max_M\{\alpha_M|\boldsymbol{\omega}_M|\})\|\boldsymbol{\omega}\|_\infty}{Ek\beta}\|g\|_0\right)
\end{aligned} \tag{5.43}$$

$$\begin{aligned}
&\Rightarrow \sum_M \alpha_M\|\kappa_M(\boldsymbol{\omega}_M\times\boldsymbol{w}_h)\|_{0,M}^2\\
&\leq \frac{1}{Ek}\left(\|h\|_{-1} + \frac{Ek + \gamma + (2+\max_M\{\alpha_M|\boldsymbol{\omega}_M|\})\|\boldsymbol{\omega}\|_\infty}{\beta}\|g\|_0\right)^2\\
&\Rightarrow \sum_M \alpha_M\|\kappa_M(\boldsymbol{\omega}_M\times\boldsymbol{u}_h)\|_{0,M}^2\\
&\lesssim \sum_M \alpha_M\|\kappa_M(\boldsymbol{\omega}_M\times\boldsymbol{u}_0)\|_{0,M}^2 + \sum_M \alpha_M\|\kappa_M(\boldsymbol{\omega}_M\times\boldsymbol{w}_h)\|_{0,M}^2\\
&\lesssim \frac{1}{Ek}\left(\|h\|_{-1} + \frac{Ek + \gamma + (2+\max_M\{\alpha_M|\boldsymbol{\omega}_M|\})\|\boldsymbol{\omega}\|_\infty}{\beta}\|g\|_0\right)^2.
\end{aligned} \tag{5.44}$$

Using the inf-sup stability again there exists $\boldsymbol{v}_h$ such that

$$\|\nabla \boldsymbol{v}_h\| \leq \beta^{-1}\|p_h\| \qquad (\nabla\cdot\boldsymbol{v}_h, q_h) = -(p_h, q_h) \quad \forall q_h \in Q_h.$$

and we find

$$\begin{aligned}
&\beta\|\nabla \boldsymbol{v}_h\|\|p_h\| \leq \|p_h\|_0^2 = -(p_h, \nabla\cdot\boldsymbol{v}_h)\\
&\leq (h, \boldsymbol{v}_h) - Ek(\nabla\boldsymbol{u}_h, \nabla\boldsymbol{v}_h) - a_h(\boldsymbol{\omega}, \boldsymbol{u}_h, \boldsymbol{\omega}, \boldsymbol{v}_h)\\
&\quad - 2(\boldsymbol{\omega}\times\boldsymbol{u}_h, \boldsymbol{v}_h) - \gamma(\nabla\cdot\boldsymbol{u}_h, \nabla\cdot\boldsymbol{v}_h)\\
&\leq (\|h\|_{-1} + (Ek+\gamma+2\|\boldsymbol{\omega}\|)\|\nabla\boldsymbol{u}_h\|\\
&\quad + \max_M\{\alpha_M\} a_h(\boldsymbol{\omega}, \boldsymbol{u}_h, \boldsymbol{\omega}, \boldsymbol{u}_h)^{1/2}\|\boldsymbol{\omega}\|)\|\nabla\boldsymbol{v}_h\|\\
&\leq (\|h\|_{-1} + (Ek+\gamma+2\|\boldsymbol{\omega}\|)\\
&\quad \left(\frac{\|h\|_{-1}}{Ek} + \frac{Ek+\gamma+(2+\max_M\{\alpha_M|\boldsymbol{\omega}_M|\})\|\boldsymbol{\omega}\|_\infty}{Ek\beta}\|g\|_0\right)\\
&\quad + \frac{\max_M\{\alpha_M|\boldsymbol{\omega}_M|\}}{\sqrt{Ek}}\\
&\quad \left(\|h\|_{-1} + \frac{Ek+\gamma+(2+\max_M\{\alpha_M|\boldsymbol{\omega}_M|\})\|\boldsymbol{\omega}\|_\infty}{\beta}\|g\|_0\right))\|\nabla\boldsymbol{v}_h\|\\
\Rightarrow \|p_h\| &\leq \frac{C}{\beta}\left(1 + \frac{Ek+\gamma+2\|\boldsymbol{\omega}\|}{Ek} + \frac{\max_M\{\alpha_M|\boldsymbol{\omega}_M|\}}{\sqrt{Ek}}\right)\|h\|_{-1}\\
&\quad + \frac{C}{\beta}\frac{Ek+\gamma+(2+\max_M\{\alpha_M|\boldsymbol{\omega}_M|\})\|\boldsymbol{\omega}\|_\infty}{Ek\beta}\\
&\quad \left(Ek+\gamma+2\|\boldsymbol{\omega}\| + \sqrt{Ek}\max_M\{\alpha_M|\boldsymbol{\omega}_M|\}\right)\|g\|_0.
\end{aligned}$$

Applying this result to the previous inhomogeneous Stokes problem (5.37) yields

$$\sum_{n=0}^N \|\nabla\widetilde{\boldsymbol{e}}_u^n\|^2 + \sum_{n=0}^N \|e_p^n - Ek\pi_p\nabla\cdot\widetilde{\boldsymbol{e}}_u^n\|^2 \leq \frac{C}{Ek^2}(\Delta t)^3. \tag{5.45}$$

Finally, we note

$$\sum_{n=0}^N \|e_p^n\|^2 \lesssim \sum_{n=0}^N \left(\|e_p^n - Ek\pi_p\nabla\cdot\widetilde{\boldsymbol{e}}_u^n\|^2 + \|Ek\pi_p\nabla\cdot\widetilde{\boldsymbol{e}}_u^n\|^2\right) \leq \frac{C}{Ek^2}(\Delta t)^3.$$

□

Now, we derived all error estimates due to time discretization that we wanted and collect them in the following theorem.

THEOREM 5.1: *The errors due to time discretization converge according to*

$$\|\boldsymbol{e}_u\|^2_{l^2(0,T;L^2(\Omega))} + \Delta t\left(\|\boldsymbol{e}_u\|^2_{l^2(0,T;LPS)} + \|e_p\|^2_{l^2(0,T;L^2(\Omega))}\right) \leq C(\Delta t)^4. \tag{5.46}$$

Proof: Lemma 5.5 and Lemma 5.3. □

6. Numerical Examples

For numerical simulations, we take advantage of the C++-FEM package `deal.ii` [16]. Let us first summarize some numerical experience with "academic" numerical examples in inertial frames of reference, see [1, 17]:

- For most the "academic" we see a relevant dependence of the error w.r.t to the velocity error in case of *one-level methods* as in Subsec. 4.1-4.2. Only when separation occurs or we consider non-convex domains an effect of LPS-SU stabilization can be observed [1]. In particular, for a parameter choice due to $\tau_M \sim h/\|\boldsymbol{u}\|_M$ best results are achieved for the energy cascade in a decaying, homogeneous, isotropic turbulence.
- The *two-level approach* is applied in [17] for methods with compatibility condition which are covered by the theory in Subsec. 4.2. For an academic example and the two-dimensional driven cavity problem with Reynolds numbers $Re_\Omega \in \{1000, 7500\}$, similar conclusions as for the one-level method are found. In particular, for the driven cavity problem a very good agreement with benchmark results on much finer grids is observed.
- For results concerning the parameter design w.r.t. to the time discretization we again refer to [15]. We basically see no influence for the local projection stabilization in inertial frames of references. However, the grad-div stabilization parameter improves the velocity error for higher Reynolds numbers much. On the other hand best results for the pressure are achieved when no grad-div stabilization is used. For lower Reynolds number we see a tremendous effect of the rotational correction in the projection step.

6.1. Analytical Reference

We first consider a manufactored solution, such that we know a solution we can compare to and evaluate the rates of convergence.
Choose $\boldsymbol{f}$ such that the following pair is a solution in $\Omega = [0,1]^2$:

$$\boldsymbol{u}(x) = \sin(\pi t)\left(-\cos\left(\frac{1}{2}\pi x\right)\sin\left(\frac{1}{2}\pi y\right), \sin\left(\frac{1}{2}\pi x\right)\cos\left(\frac{1}{2}\pi y\right)\right)^T$$

$$p(x) = -\pi \sin\left(\frac{1}{2}\pi x\right)\sin\left(\frac{1}{2}\pi y\right)\sin(\pi t).$$

In this example we use the stabilization parameters $\gamma = 1$, $\tau_M = 1/|\boldsymbol{u}_M|^2$ and $\alpha_M = 1$. The time step size is fixed to $\Delta t = 10^{-3}$. Figures 1 and 2 show the effect of the stabilization for various Ekman numbers, $\boldsymbol{\omega} = (0,0,1)^T$ and $Ro = 1$.

We clearly see that the order of convergence in the unstabilized case deterioates quickly with decreasing Ekman number. Using stabilization we acquire the expected rates of convergence. Just for the finest meshes we see that the time step size is dominating the error. For the pressure the error is independent of stabilization and Ekman number. In this example the effect of LPS stabilization is negligible (not shown).

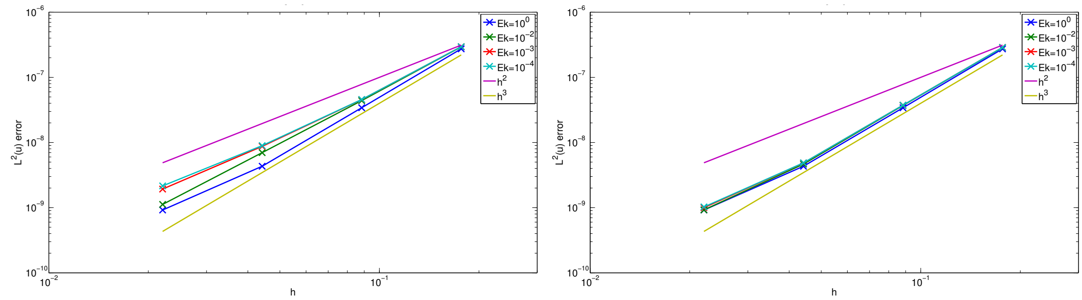

Figure 1: Velocity error w.r.t. $L^2(\Omega)$ for the analytical testcase, unstabilized (left) and stabilized (right)

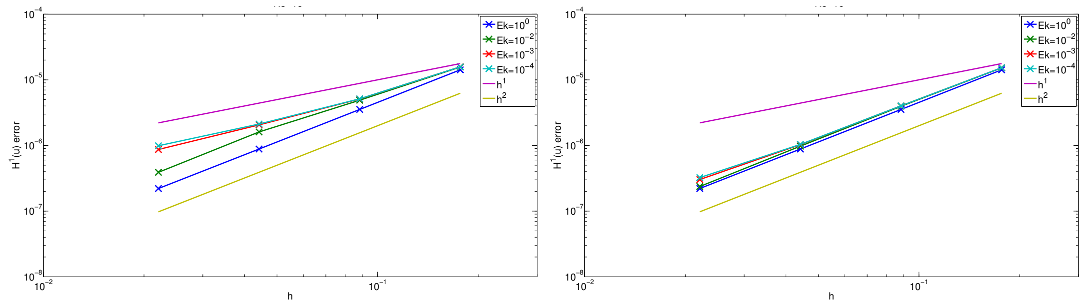

Figure 2: Velocity error w.r.t. $H^1(\Omega)$ for the analytical testcase, unstabilized (left) and stabilized (right)

6.2. Rotating Poiseuille Flow

Next we turn our attention to a slightly more realistic case. We consider a channel given by the domain $\Omega = [-2, 2] \times [-1, 1]$ which rotates around its midpoint and the inflow is given by a quadratic profile.

$$\boldsymbol{u}(x, y) = \begin{cases} (1 - y^2, 0)^T, & x = -2 \\ (0, 0)^T, & |y| = 1 \end{cases}$$

$$(\nabla \boldsymbol{u} \cdot \boldsymbol{n})(x = 2, y) = 0$$

$$\boldsymbol{u}_0 = 0, \quad p_0 = 0, \quad \boldsymbol{f} = 0$$

For the critical parameters we choose $\boldsymbol{\omega} = (0, 0, 100)^T$ and $Ek = 10^{-3}$. The basic flow we expect is one where all outflow happens in a small area on the bottom left side. In particular, the streamlines are strongly curved at the outflow boundary and resolving the boundary layers there by stabilization or grid refinement is important to prevent that oscillations occur.

For this example we use the stabilization parameters $\gamma = 1$, $\tau_M = 1/|\boldsymbol{u}_M|^2$ and $\alpha_M = 1/h$. Using only grad-div stabilization leads to high oscillations that spread from the outflow into the interior of the domain (Figure 3). This behav-

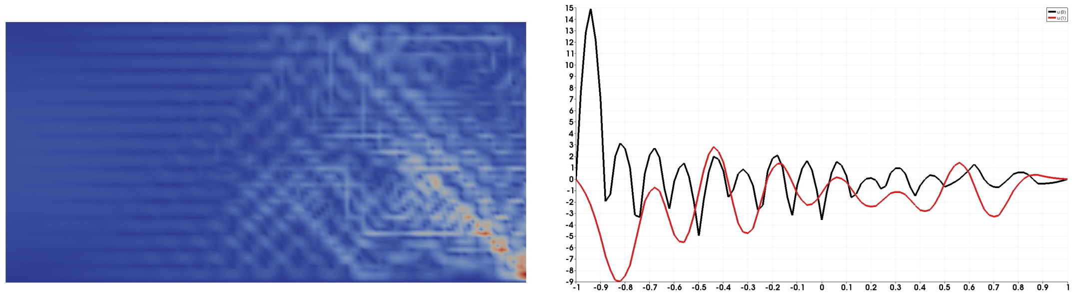

Figure 3: Rotating Poiseuille Flow, grad-div
left: Streamlines; right: Profile at the outflow boundary $x = 2$

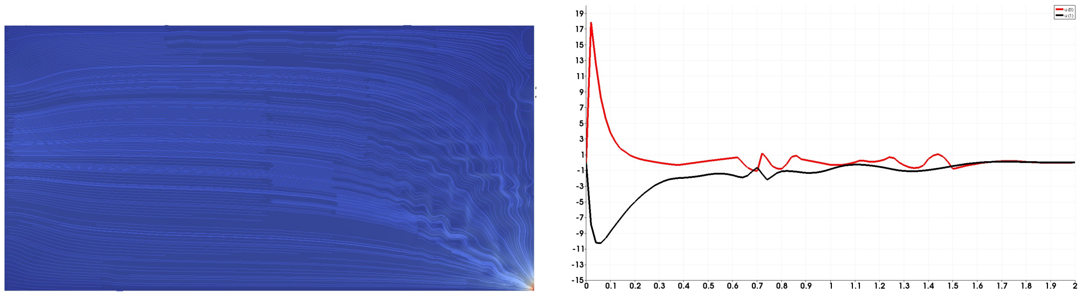

Figure 4: Rotating Poiseuille Flow, grad-div adaptive
left: Streamlines; right: Profile at the outflow boundary $x = 2$

ior improves when the mesh is refined adaptively (Figure 4), but nevertheless oscillations occur. Hence, grad-div stabilization is not sufficient in this example.

Using LPS-Coriolis stabilization additionally improves the solution a lot. All the oscillation in the interior of the domain are damped away and only at the outflow boundary smaller oscillations occur (Figure 5). If we use the LPS-SUPG stabilization instead the situation is similar but there are oscillations that spread into the interior (Figure 6).

Finally, we combine all the considered stabilizations and use adaptive mesh refinement. This finally leads to a solution that has all the features (Figure 7) and we see that all these parts are necessary.

6.3. The Proudman-Stewartson Problem

We now come to a more realistic case in which we consider the fluid motion between two rotating spheres. The frame of reference we choose is one in which the one does not move. Given that the inner cylinder rotates with a angular velocity vector $\omega_i \boldsymbol{e}_z$ and the outer one with $\omega_o \boldsymbol{e}_z$ the problem that we are solving can be stated as

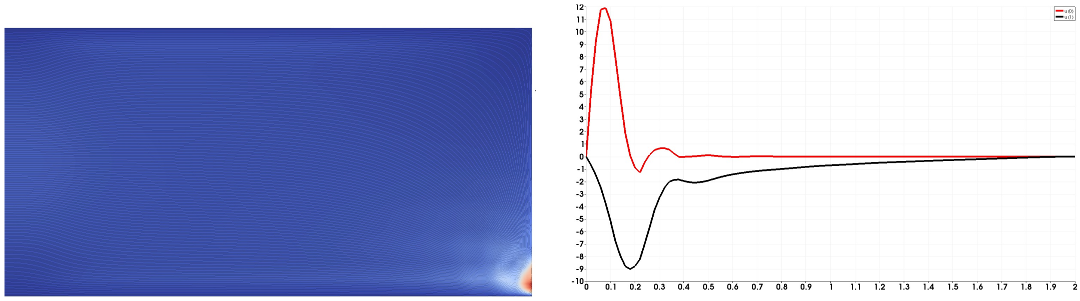

Figure 5: Rotating Poiseuille Flow, Coriolis
left: Streamlines; right: Profile at the outflow boundary $x = 2$

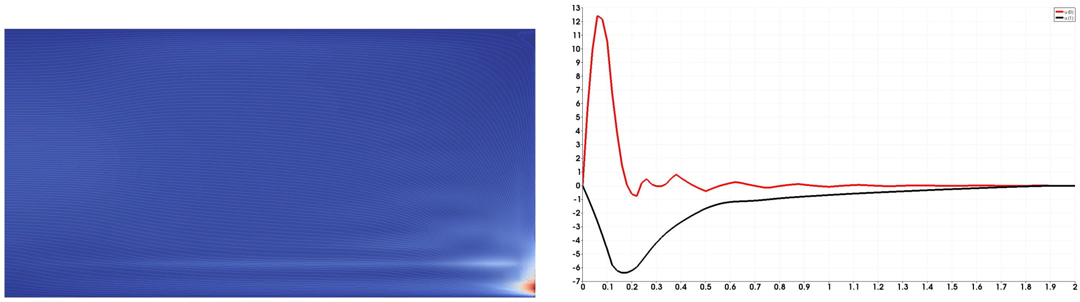

Figure 6: Rotating Poiseuille Flow, SUPG
left: Streamlines; right: Profile at the outflow boundary $x = 2$

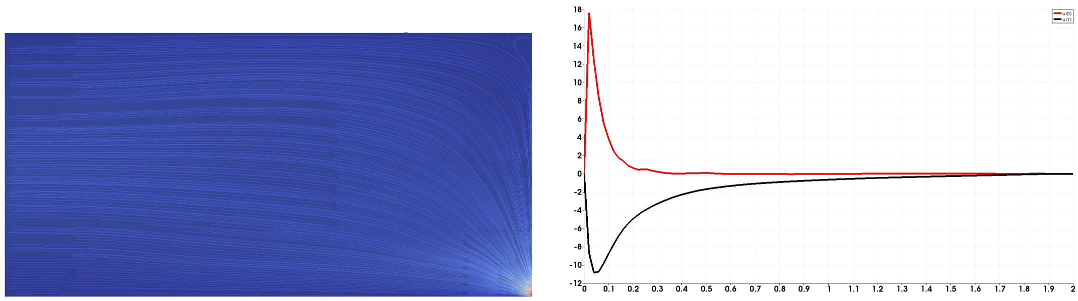

Figure 7: Rotating Poiseuille Flow, SUPG Coriolis Adaptive
left: Streamlines; right: Profile at the outflow boundary $x = 2$

$$
\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} + Ro(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + 2\hat{\boldsymbol{e}}_z \times \boldsymbol{u} &= Ek\Delta\boldsymbol{u} - \nabla p \\
\nabla\cdot\boldsymbol{u} &= 0 \\
\boldsymbol{u} &= r\sin\theta\hat{\boldsymbol{e}}_\phi \qquad \text{at } r = r_i \\
\boldsymbol{u} &= \boldsymbol{0} \qquad \text{at } r = r_o
\end{aligned}
$$

where the critical parameters are defined as follows

$$
\begin{aligned}
r_i &= 1/2 & r_o &= 3/2 \\
Ro &:= \frac{\omega_i - \omega_o}{\omega_o} & Ek &:= \frac{\nu}{\omega_o(r_o - r_i)^2}.
\end{aligned}
$$

The flow that we expect to see is one in which the angular velocity is between the ones on the outer and inner sphere in the cylinder $r < r_i$. Outside this cylinder the flow should basically be at rest. For small Ekman numbers we expect to see basically a solution that is constant in z-direction and following the motion of the inner sphere, i.e.

$$\boldsymbol{u} = r\sin\theta\hat{\boldsymbol{e}}_\phi$$

in the rotating frame of reference. The boundary layers (Ekman layers) at the inner and outer sphere have a width according to $Ek^{1/2}$. A secondary flow is given by a meridional circulation from the outer to the inner Ekman layer inside the cylinder $r < r_i$ and from the inner to the outer Ekman layer outside this cylinder.

In order to resolve the various flow structure we again take advantage of an adaptive mesh refinement that is based on the jump of the gradient of the solution along the faces of each cell. We expect to see that most of refinement takes place near the inner and outer sphere and the boundary of the tangent cylinder, i.e. at $r = r_i$. A typical picture can be seen in Figure 8.

In the following Figures 9 - 14 we consider the occurring flow different Ekman and Rossby numbers. In agreement with [18] we observe only minor effects for the profile of the angular velocity with respect to Ro. However, the solutions for $Ek = 4.5$ are unstable and thus not shown here. Apart from that we see that the chosen parameter setting in combination with the use of adaptive mesh refinement is sufficient to resolve all the relevant flow features.

6.4. Precessing Sphere

The last example that we are going to consider is a precessing sphere. That is work in progress. At the moment we are interested in confirming the results that Y.Lin, P.Marti and J.Noir in [19] obtained for precessional instabilities in

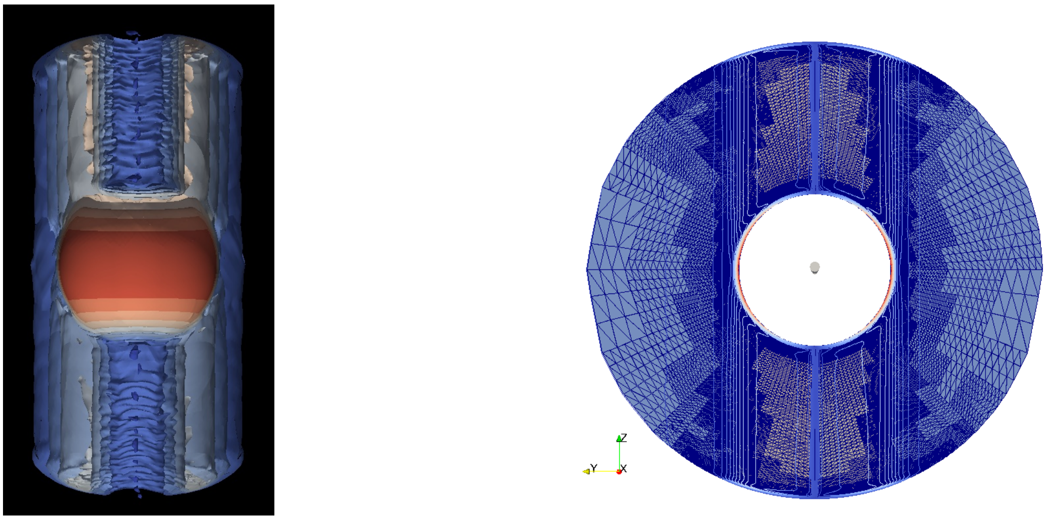

Figure 8: Proudman-Stewartson problem,
left: Iso faces of the velocity magnitude for $Ek = 10^{-6}, Ro = 0$;
right: Mesh and isolines for $Ek = 10^{-4}, Ro = .5$

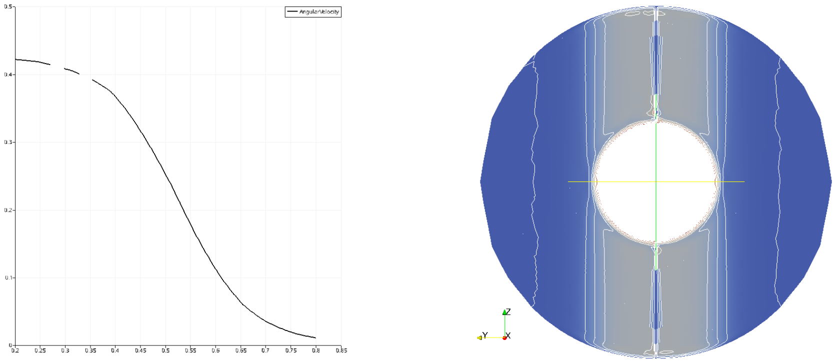

Figure 9: Instabilities of the Stewartson Layer, Ek=$10^{-3.5}$, Ro=$-.5$

a rotating spherical cavity. Considering the frame in which the mantle frame is fixed the set of equations that we solve for this problem is given by

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - Ek\Delta \boldsymbol{u} + 2(\hat{\boldsymbol{k}} + Po\hat{\boldsymbol{k}}_p) \times \boldsymbol{u} = \boldsymbol{f}_{Po} - \nabla p$$
$$\nabla \cdot \boldsymbol{u} = 0$$
$$\boldsymbol{u} = \boldsymbol{0} \text{ in } \partial\Omega \times (0, T]$$

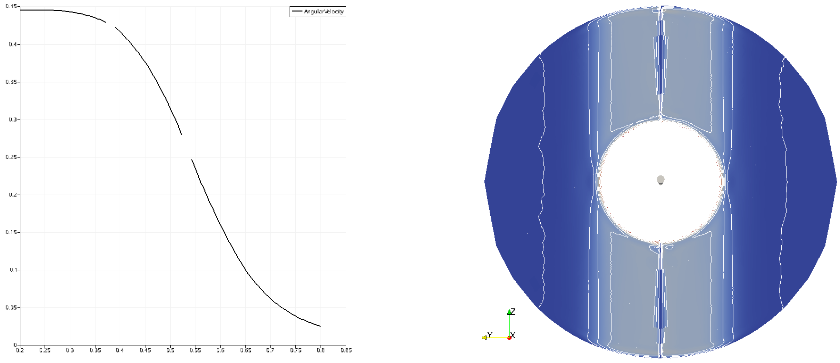

Figure 10: Instabilities of the Stewartson Layer, Ek=$10^{-3.5}$, Ro=0

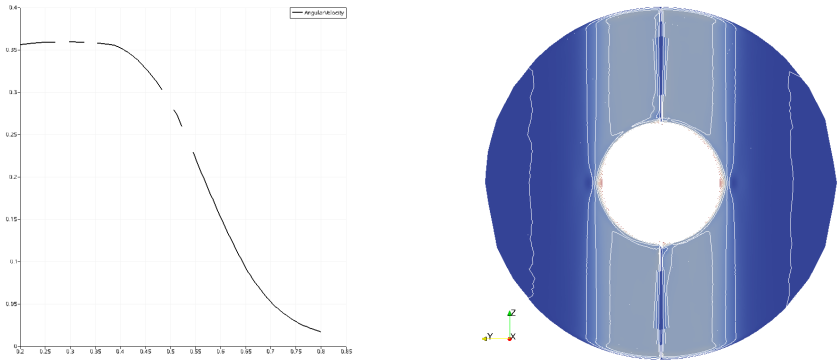

Figure 11: Instabilities of the Stewartson Layer, Ek=$10^{-3.5}$, Ro=.5

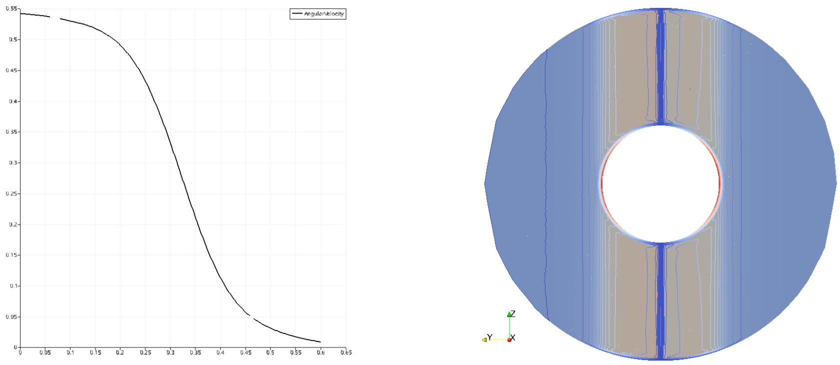

Figure 12: Instabilities of the Stewartson Layer, Ek=10^{-4}, Ro=−.5

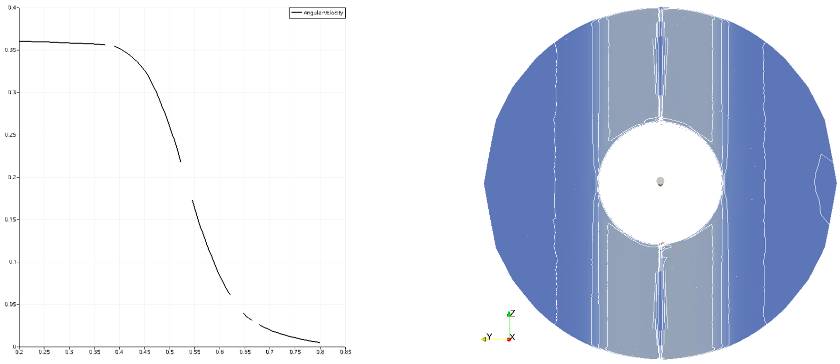

Figure 13: Instabilities of the Stewartson Layer, Ek=10^{-4}, Ro=0

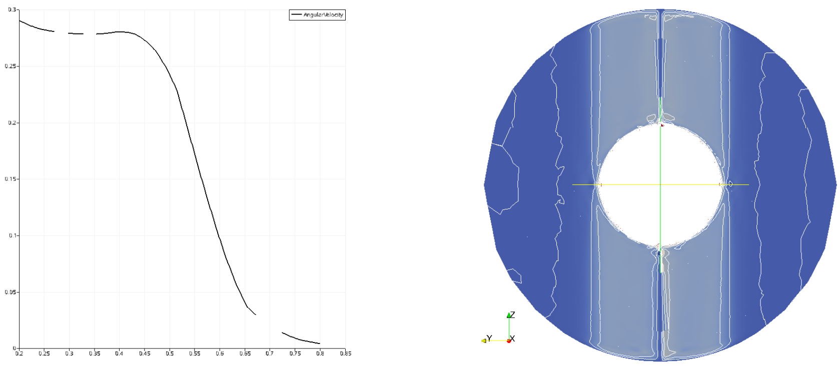

Figure 14: Instabilities of the Stewartson Layer, Ek=10^{-4}, Ro=.5

where the Poincar force is given by $\boldsymbol{f}_{Po} = Po(\hat{\boldsymbol{k}}_p \times \hat{\boldsymbol{k}}) \times \boldsymbol{r}$

$$\begin{aligned} \hat{\boldsymbol{k}}_p &= (\hat{\boldsymbol{\imath}} \cos(t) - \hat{\boldsymbol{\jmath}} \sin(t)) \sin(\alpha_p) + \hat{\boldsymbol{k}} \cos(\alpha_p) \\ Ek &:= \frac{Ek}{\Omega_o R^2} \quad Po := \frac{\Omega_P}{\Omega_o}. \end{aligned}$$

At first we are interested in the main flow that is excited by the precessional force. Therefore we consider the angle that the main fluid rotation axis forms with the rotation axis around which the precession axis rotates. Defining the velocity in the pression frame by

$$\boldsymbol{u}^p = \boldsymbol{u} + \hat{\boldsymbol{z}} \times \boldsymbol{r}$$

this angle α_F is defined according to

$$\begin{aligned} 2\boldsymbol{\omega}_F &= \langle \nabla \times \boldsymbol{u}^P \rangle = \langle \nabla \times \boldsymbol{u} \rangle + 2\hat{\boldsymbol{z}} \\ \cos(\alpha_F) &= \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{\omega}}_F \end{aligned}$$

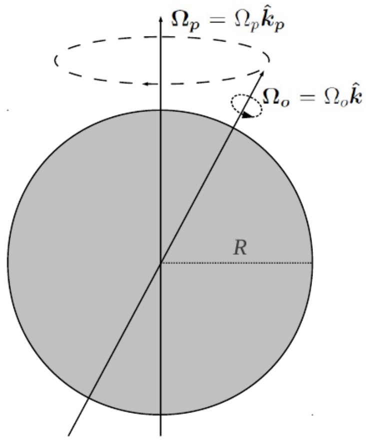

Figure 15: Y.Lin, P.Marti, J.Noir: Sketch of the problem

Table 1: The angle α_F between rotation axes of the container and the fluid in dependence of the Poincare number and the Ekman number

Ek	10^{-4}	10^{-5}	10^{-6}	$3 \cdot 10^{-5}$	$3 \cdot 10^{-5}$	$3 \cdot 10^{-5}$
Po	-10^{-4}	-10^{-4}	-10^{-4}	-10^{-3}	$-.0007$	-0.014
α_F	0.0014	0.0099	0.0029	0.020	0.49	0.32

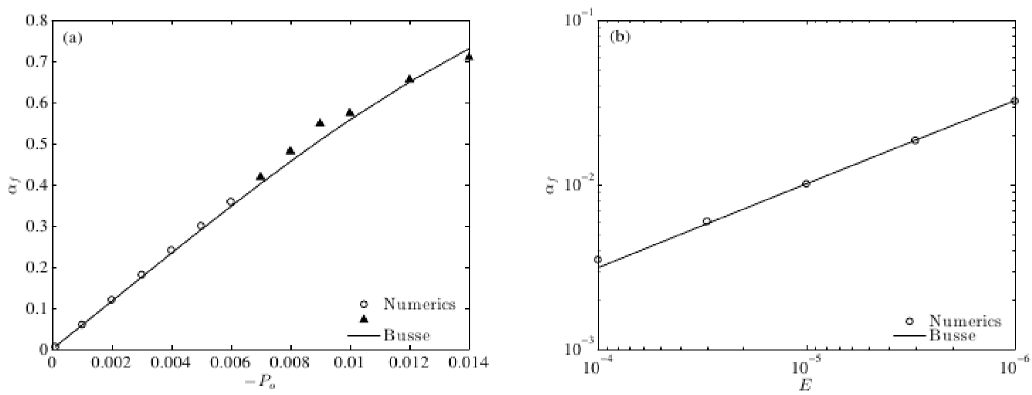

Figure 16: left: α_F in dependence on Po at fixed $Ek = 3.0 \times 10^{-5}$;
right: α_F in dependence on Ek at fixed $Po = -1.0 \times 10^{-4}$

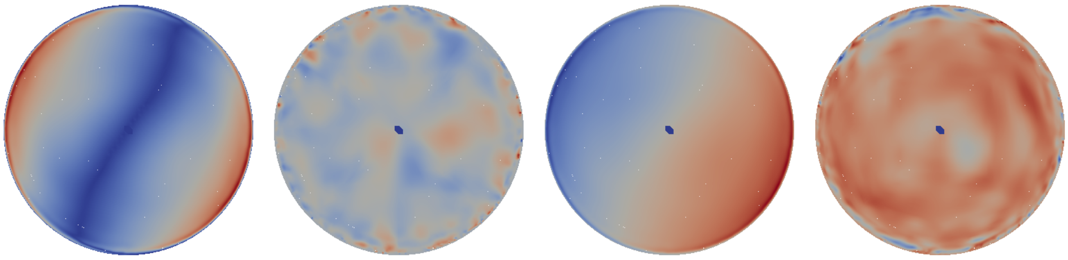

Figure 17: Precessing Sphere, $Ek = 10^{-6}, Po = -10^{-4}$,
View on the equatorial plane, left to right: $\|u\|, u_r, u_\theta, u_\phi$

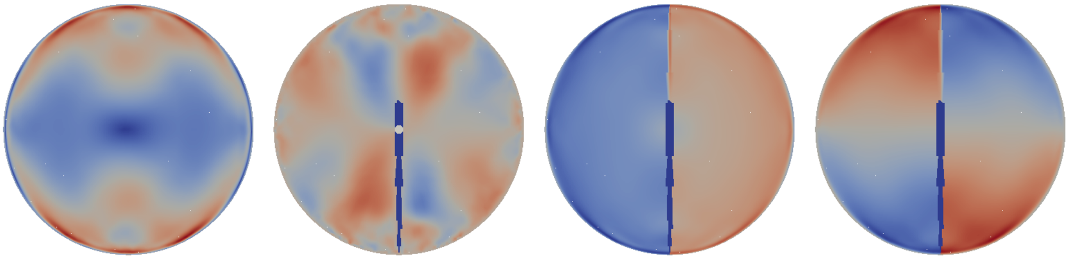

Figure 18: Precessing Sphere, $Ek = 10^{-6}, Po = -10^{-4}$,
View on the $x = 0$ plane (z-axis up), left to right: $\|u\|, u_r, u_\theta, u_\phi$

Our results with respect to this quantity can be seen in Table 1. Comparing with the diagram that Lin et al. obtained (see Figure 16) we observe a good agreement with the left plot up to $Po = 0.007$ and with the right plot up to $Po = 10^{-5}$. We suspect the deviation of the other points to be a consequence of averaging the vorticity over too much of the near boundary area.

Exemplarily, we show in Figures 17 and 18 the flow patterns for $Ek = 10^{-6}$, $Po = -10^{-4}$ that we observe after substracting the main fluid rotation. Due to the fact that this pictures show a sufficient resolution of the boundary layer we see our previous suspection for the computation of the main fluid rotation axis confirmed.

7. Discussion and Summary

We considered conforming finite element (FE) approximations of the time-dependent Navier-Stokes problem with inf-sup stable approximation of velocity and pressure in rotating frames of reference. We introduced a variant of the local projection stabilization method for dealing with cases in which critical parameter introduce unphysical oscillations to the solution. The approach combines ideas of streamline upwinding, grad-div stabilization and stabilization of theskew-symmetric coriolis term.

A stability and convergence analysis is provided for the arising nonlinear semidiscrete problem. Similar to [9] and our observations in [15], we can show that the Gronwall constant does not explicitly depend on the Reynolds number Re_Ω for velocities $\boldsymbol{u} \in [L^\infty(0,T;W^{1,\infty}(\Omega)]^d$. In the interesting case of LPS methods without an additional compatibility condition between the basic local velocity space and the projection space, our approach improves a result of Matthies/Tobiska in [7] for the Oseen problem. If the mentioned compatibility condition is valid, we can remove a restriction on the local mesh width which appeared in the former case.

The grad-div stabilization with parameters $\gamma \sim 1$ seems to be essential for improved mass conservation and velocity estimates in $W^{1,2}(\Omega)$. Numerical examples confirm these theoretical results. In particular, for boundary layer flows the SUPG-type stabilization $\tau_M \sim 1/\boldsymbol{u}_M^2$ seems to be important for modeling unresolved velocity scales. However, in case of dominating rotation a stabilization that only affects the streamline direction of the flow does not seem to be sufficient. In this case the suggested stabilization of the coriolis term is essential. Furthermore, the results show that the proposed approach is capable of resolving flow structures in physically interesting cases.

Future considerations to further examine the flow structures in spherical precessing domains and thus confirming the results from ETH Zürich group. Furthermore we want to extend the observations to ellipsoidal domains with a small eccentricity.

A. The Inverse Stokes Operator

For the defined ansatz spaces $\boldsymbol{V}_h$ and Q_h we define the (grad-div and Coriolis stabilized) inverse Stokes operator as the solution $S\boldsymbol{v} \in \boldsymbol{V}_h$ of the problem

$$\begin{aligned} Ek(\nabla S\boldsymbol{v}, \nabla \boldsymbol{w}) - (r, \nabla \cdot \boldsymbol{w}) + 2(\boldsymbol{\omega} \times S\boldsymbol{v}, \boldsymbol{w}) & \\ +\gamma(\nabla \cdot S\boldsymbol{v}, \nabla \cdot \boldsymbol{w}) + a_h(\boldsymbol{\omega}, S\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{w}) = (\boldsymbol{v}, \boldsymbol{w}) & \qquad \forall \boldsymbol{w} \in \boldsymbol{V}_h \\ (\nabla \cdot S\boldsymbol{v}, q) = 0 & \qquad \forall q \in Q_h \end{aligned}$$

In particular $S\boldsymbol{v}$ is discretely solenoidal, i.e. $S\boldsymbol{v} \in \boldsymbol{V}_h^{div}$.

By testing this equation symmetrically we can derive an estimate on the solution in the H^1-Seminorm

$$\begin{aligned} & Ek\|\nabla S\boldsymbol{v}\|_0^2 + \gamma\|\nabla \cdot S\boldsymbol{v}\|_0^2 + \sum_M \left(\alpha_M \|\kappa_M(\boldsymbol{\omega} \times S\boldsymbol{v})\|_{0,M}^2\right) \\ & \quad = (\boldsymbol{v}, S\boldsymbol{v}) \leq \|\boldsymbol{v}\|_{-1}\|\nabla S\boldsymbol{v}\| \\ & \Rightarrow \|\nabla S\boldsymbol{v}\| \leq \frac{1}{Ek}\|\boldsymbol{v}\|_{-1} \\ & \Rightarrow \sum_M \alpha_M \|\kappa_M(\boldsymbol{\omega} \times S\boldsymbol{v})\|_{0,M}^2 \leq \frac{1}{Ek}\|\boldsymbol{v}\|_{-1}^2. \end{aligned} \tag{A.1}$$

According to the discrete inf-sup condition (2.3), we have for $r \in Q_h$ the existence of a unique $\boldsymbol{w} \in \boldsymbol{V}_h$ with

$$\begin{aligned}&(\nabla \cdot \boldsymbol{w}, q) = -(r, q) \quad \forall q \in Q_h\\ &\|\nabla \boldsymbol{w}\| \leq \beta^{-1}\|r\|.\end{aligned}$$

Testing with $(\boldsymbol{w}, 0) \in \boldsymbol{V}_h \times Q_h$, we obtain

$$\begin{aligned}&\beta\|\nabla \boldsymbol{w}\|\|r\| \leq \|r\|_0^2\\ &\leq (\boldsymbol{v}, \boldsymbol{w}) - Ek(\nabla S\boldsymbol{v}, \nabla \boldsymbol{w}) - \gamma(\nabla \cdot S\boldsymbol{v}, \nabla \cdot \boldsymbol{w})\\ &\quad - a_h(\boldsymbol{\omega}, S\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{w}) - 2(\boldsymbol{\omega} \times S\boldsymbol{v}, \boldsymbol{w})\\ &\leq (\|\boldsymbol{v}\|_{-1} + (Ek + \gamma)\|\nabla S\boldsymbol{v}\|\\ &\quad + C_P \max_M\{\sqrt{\alpha_M}\} a_h(\boldsymbol{\omega}, S\boldsymbol{v}, \boldsymbol{\omega}, S\boldsymbol{v})^{1/2}\|\boldsymbol{\omega}\|\|\nabla \boldsymbol{w}\|\\ &\quad + 2C_p^2\|\boldsymbol{\omega}\|\|\nabla S\boldsymbol{v}\|\|\nabla \boldsymbol{w}\|\\ &\leq \left(2 + \frac{\gamma}{Ek} + \left(C_P \frac{\max_M\{\sqrt{\alpha_M}\}}{\sqrt{Ek}} + \frac{2C_p^2}{Ek}\right)\|\boldsymbol{\omega}_M\|\right)\|\boldsymbol{v}\|_{-1}\|\nabla \boldsymbol{w}\|.\end{aligned}$$

A combination of these estimates states

$$\|r\| + Ek\|\nabla S\boldsymbol{v}\| \leq C\left(1 + \frac{\gamma}{Ek} + \left(\frac{\max_M\{\sqrt{\alpha_M}\}}{\sqrt{Ek}} + \frac{1}{Ek}\right)\|\boldsymbol{\omega}_M\|\right)\|\boldsymbol{v}\|_{-1}. \tag{A.2}$$

Provided the solution is sufficiently smooth we test with $(-\Delta S\boldsymbol{v}, -\Delta r)$ to get

$$\begin{aligned}&Ek\|\Delta S\boldsymbol{v}\|_0^2 + \gamma\|\nabla\nabla \cdot S\boldsymbol{v}\|_0^2 + \alpha\|\boldsymbol{\omega} \times \nabla S\boldsymbol{v}\|_0^2\\ &= Ek(\nabla \cdot \nabla S\boldsymbol{v}, \nabla \cdot \nabla S\boldsymbol{v}) + \gamma(\nabla\nabla \cdot S\boldsymbol{v}, \nabla\nabla \cdot S\boldsymbol{v}) + a_h(\boldsymbol{\omega}, \nabla S\boldsymbol{v}, \boldsymbol{\omega}, \nabla S\boldsymbol{v})\\ &= -Ek(\nabla S\boldsymbol{v}, \nabla\Delta S\boldsymbol{v}) + (r, \nabla \cdot \Delta S\boldsymbol{v}) - (\nabla \cdot S\boldsymbol{v}, \Delta r) - 2(\boldsymbol{\omega} \times S\boldsymbol{v}, \Delta S\boldsymbol{v})\\ &\quad - a_h(\boldsymbol{\omega}, S\boldsymbol{v}, \boldsymbol{\omega}, \Delta S\boldsymbol{v}) - \gamma(\nabla \cdot S\boldsymbol{v}, \nabla \cdot \Delta S\boldsymbol{v})\\ &= -(\boldsymbol{v}, \Delta S\boldsymbol{v}) \leq \|\boldsymbol{v}\|\|\Delta S\boldsymbol{v}\|\\ &\Rightarrow \|\Delta S\boldsymbol{v}\| \leq \frac{1}{Ek}\|\boldsymbol{v}\|\\ &\Rightarrow \sum_M \alpha_M\|\kappa_M(\boldsymbol{\omega}_M \times \nabla S\boldsymbol{v})\|_{0,M}^2 \leq \frac{1}{Ek}\|\boldsymbol{v}\|^2\end{aligned} \tag{A.3}$$

For the pressure we get by testing with $\boldsymbol{w} = \nabla r$

$$\begin{aligned}
\|\nabla r\|_0^2 &= -(r, \nabla\cdot\nabla r)\\
&= -Ek(\nabla S\boldsymbol{v}, \nabla\nabla r) - \gamma(\nabla\cdot S\boldsymbol{v}, \nabla\cdot\nabla r) + (\boldsymbol{v}, \nabla r)\\
&\quad - a_h(\boldsymbol{\omega}, S\boldsymbol{v}, \boldsymbol{\omega}, \nabla r) - 2(\boldsymbol{\omega}\times S\boldsymbol{v}, \nabla r)\\
&= Ek(\Delta S\boldsymbol{v}, \nabla r) + \gamma(\nabla\nabla\cdot S\boldsymbol{v}, \nabla r) + (\boldsymbol{v}, \nabla r)\\
&\quad - a_h(\boldsymbol{\omega}, S\boldsymbol{v}, \boldsymbol{\omega}, \nabla r) - 2(\boldsymbol{\omega}\times S\boldsymbol{v}, \nabla r)\\
&\leq (Ek\|\Delta S\boldsymbol{v}\| + \gamma\|\nabla\nabla\cdot S\boldsymbol{v}\| + \|\boldsymbol{v}\|)\|\nabla r\|\\
&\quad - a_h(\boldsymbol{\omega}, S\boldsymbol{v}, \boldsymbol{\omega}, \nabla r) - 2(\boldsymbol{\omega}\times S\boldsymbol{v}, \nabla r)\\
&\leq ((Ek+\gamma)\|\Delta S\boldsymbol{v}\| + \|\boldsymbol{v}\|)\|\nabla r\|\\
&\quad + \max_M\{\sqrt{\alpha_M}\} a_h(\boldsymbol{\omega}, S\boldsymbol{v}, \boldsymbol{\omega}, S\boldsymbol{v})\|\boldsymbol{\omega}\|\|\nabla r\| + 2\|\boldsymbol{\omega}\|\|S\boldsymbol{v}\|\|\nabla r\|\\
&\leq \left(2 + \frac{\gamma}{Ek} + \left(\frac{\max_M\{\sqrt{\alpha_M}\}}{\sqrt{Ek}} + \frac{2}{Ek}\right)\|\boldsymbol{\omega}_M\|\right)\|\boldsymbol{v}\|\|\nabla r\|\\
\Rightarrow \|\nabla r\| &\leq \left(2 + \frac{\gamma}{Ek} + \left(\frac{\max_M\{\sqrt{\alpha_M}\}}{\sqrt{Ek}} + \frac{2}{Ek}\right)\|\boldsymbol{\omega}_M\|\right)\|\boldsymbol{v}\|
\end{aligned}$$

using the vector identity $\nabla\times\nabla\times\boldsymbol{v} = \nabla\nabla\cdot\boldsymbol{v} - \Delta\boldsymbol{v}$ and $(\nabla\times\nabla\times\boldsymbol{v}, \nabla\nabla\cdot\boldsymbol{v}) = 0$.

Next we are interested in a lower bound for the seminorm induced by the inverse Stokes operator.

$$\begin{aligned}
|\boldsymbol{v}|_* :=& Ek(\nabla S\boldsymbol{v}, \nabla\boldsymbol{v}) + \gamma(\nabla\cdot S\boldsymbol{v}, \nabla\cdot\boldsymbol{v}) + a_h(\boldsymbol{\omega}, S\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{v})\\
=&\|\boldsymbol{v}\|_0^2 + (r, \nabla\cdot\boldsymbol{v})\\
=&\|\boldsymbol{v}\|_0^2 - (\nabla r, \boldsymbol{v}-\boldsymbol{v}^*) \quad \forall\boldsymbol{v}^*\in\boldsymbol{V}_h^{div}\\
\geq&\|\boldsymbol{v}\|_0^2 - \|\nabla r\|\|\boldsymbol{v}-\boldsymbol{v}^*\|\\
\geq&\|\boldsymbol{v}\|_0^2 - \left(2 + \frac{\gamma}{Ek} + \left(\frac{\max_M\{\sqrt{\alpha_M}\}}{\sqrt{Ek}} + \frac{2}{Ek}\right)\|\boldsymbol{\omega}_M\|\right)\|\boldsymbol{v}\|\|\boldsymbol{v}-\boldsymbol{v}^*\| \qquad \text{(A.4)}\\
\geq&\left(1 - \left(2 + \frac{\gamma}{Ek} + \left(\frac{\max_M\{\sqrt{\alpha_M}\}}{\sqrt{Ek}} + \frac{2}{Ek}\right)\|\boldsymbol{\omega}_M\|\right)^2\frac{\epsilon}{4}\right)\|\boldsymbol{v}\|_0^2\\
&- \frac{1}{\epsilon}\|\boldsymbol{v}-\boldsymbol{v}^*\|_0^2 \quad \forall\epsilon > 0
\end{aligned}$$

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