# Finite Element Methods for Flow Simulations

Daniel Arndt

Heidelberg University

Interdisciplinary Center for Scientific Computing

The Heidelberg Laureate Forum at MATCH

21. September 2016

# Overview

# Incompressible Navier-Stokes Equations

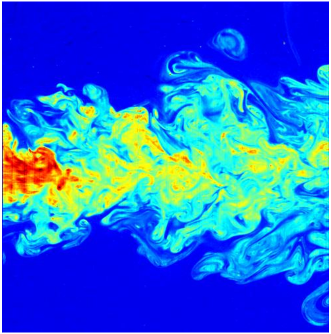
## Momentum Equation - Inertial Frame of Reference

$$\partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f}_u$$
$$\nabla \cdot \boldsymbol{u} = 0$$

Motion of viscous, incompressible fluids

- $\boldsymbol{u}$ velocity
- $p$ velocity
- $\nu$ kinematic viscosity

- Parabolic PDE of second order
- Non-linear
- Challenge: $\nu \to 0$



C. Fukushima and J. Westerweel, Technical University of Delft (Wikimedia Commons)

# Navier-Stokes Equations - Millenium Problem[1]

## Momentum Equation - Inertial Frame of Reference

$$\partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f}_u \quad (1)$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad (2)$$

Open question: Existence of a solution satisfying

- (1), (2)
- $\boldsymbol{u}, p \in C^\infty(\mathbb{R}^3 \times [0, \infty))$
- $\boldsymbol{f}_u \equiv \boldsymbol{0}$
- $\int_{\mathbb{R}^n} |\boldsymbol{u}(\boldsymbol{x}, t)|^2 \, dx \leq C \quad \forall t \geq 0$

for a given $\boldsymbol{u}(\boldsymbol{x}, 0) \in C^\infty(\mathbb{R}^3)$, $\nabla \cdot \boldsymbol{u}(\boldsymbol{x}, 0) = 0$.

[1] Charles L Fefferman. “Existence and smoothness of the Navier-Stokes equation”. In: *The millennium prize problems* (2006), pp. 57–67

# Weak Formulation

Consider a bounded domain $\Omega$ and the function spaces

$$\boldsymbol{V} := [W_0^{1,2}(\Omega)]^d \qquad\qquad Q := L_*^2(\Omega)$$

Find $(\boldsymbol{u}, p)\colon (t_0, T) \to \boldsymbol{V} \times Q$ such that

$$\begin{aligned}(\partial_t \boldsymbol{u}, \boldsymbol{v}) + c_u(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (p, \nabla \cdot \boldsymbol{v}) &= (\boldsymbol{f}_u, \boldsymbol{v}),\\ (\nabla \cdot \boldsymbol{u}, q) &= 0\end{aligned}$$

for all $(\boldsymbol{v}, q) \in \boldsymbol{V} \times Q$ and $t \in (t_0, T)$ a.e.

$$c_u(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) := \frac{1}{2}\left[((\boldsymbol{w} \cdot \nabla)\boldsymbol{u}, \boldsymbol{v}) - ((\boldsymbol{w} \cdot \nabla)\boldsymbol{v}, \boldsymbol{u})\right]$$

Existence ✓ Uniqueness ?

# Overview

# Weak Formulation - Conforming Methods

Finite dimensional spaces

$$\boldsymbol{V}_h := [W_0^{1,2}(\Omega)]^d \qquad Q_h \subset Q := L^2_*(\Omega)$$

Find $(\boldsymbol{u_h}, p_h) \colon (t_0, T) \to \boldsymbol{V}_h \times Q_h$ such that

$$\begin{aligned}(\partial_t \boldsymbol{u}_h, \boldsymbol{v}_h) + c_u(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + \nu(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) - (p_h, \nabla \cdot \boldsymbol{v}_h) &= (\boldsymbol{f}_u, \boldsymbol{v}_h),\\ (\nabla \cdot \boldsymbol{u}_h, q_h) &= 0\end{aligned}$$

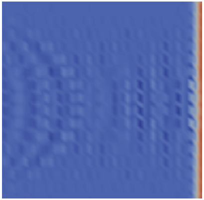
for all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times Q_h$ and $t \in (t_0, T)$ a.e.

$\to$ Nonlinear finite dimensional problem $\xrightarrow{\text{Linearize}}$ Linear algebra

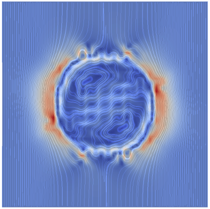
Existence ✓ Uniqueness ✓

Problem: $\nabla \cdot \boldsymbol{V}_h \not\subset Q_h \implies \nabla \cdot \boldsymbol{u}_h \neq 0$

# Problems

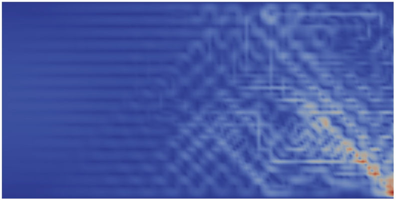


Dominant convection



Shear and boundary layers

Poor mass conservation



Large Coriolis forces
(non-inertial frame of reference)

# Stabilization

$$
\begin{aligned}
(\partial_t \boldsymbol{u}, \boldsymbol{v}) + c_u(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (p, \nabla \cdot \boldsymbol{v}) &= (\boldsymbol{f}_u, \boldsymbol{v}), \\
(\nabla \cdot \boldsymbol{u}, q) &= 0
\end{aligned}
$$

# Stabilization

$$(\partial_t \boldsymbol{u}, \boldsymbol{u}) + c_u(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{u}) + \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{u}) - (p, \nabla \cdot \boldsymbol{u}) = (\boldsymbol{f}_u, \boldsymbol{u}),$$
$$(\nabla \cdot \boldsymbol{u}, p) = 0$$

# Stabilization

$$(\partial_t \boldsymbol{u}, \boldsymbol{u}) + c_u(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{u}) + \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{u}) - (p, \nabla \cdot \boldsymbol{u}) = (\boldsymbol{f}_u, \boldsymbol{u}),$$

$$(\nabla \cdot \boldsymbol{u}, p) = 0$$

# Stabilization

$$(\partial_t \boldsymbol{u}, \boldsymbol{u}) + c_u(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{u}) + \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{u}) - (p, \nabla \cdot \boldsymbol{u}) = (\boldsymbol{f}_u, \boldsymbol{u}),$$
$$(\nabla \cdot \boldsymbol{u}, p) = 0$$

**Idea:** Stabilize only small scales

- Family of macro decompositions $\{\mathcal{M}_h\}$
- $D_M \subset [L^\infty(M)]^d$ finite elemente ansatz space on $M \in \mathcal{M}_h$.
- $\pi_M : [L^2(M)]^d \to D_M$ orthogonal $L^2$ projection,
  $\kappa_M = Id - \pi_M$ fluctuation operator

# Stabilization

$$(\partial_t \boldsymbol{u}, \boldsymbol{u}) + c_u(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{u}) + \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{u}) - (p, \nabla \cdot \boldsymbol{u}) = (\boldsymbol{f}_u, \boldsymbol{u}),$$
$$(\nabla \cdot \boldsymbol{u}, p) = 0$$

**Idea:** Stabilize only small scales

- Family of macro decompositions $\{\mathcal{M}_h\}$
- $D_M \subset [L^\infty(M)]^d$ finite elemente ansatz space on $M \in \mathcal{M}_h$.
- $\pi_M : [L^2(M)]^d \to D_M$ orthogonal $L^2$ projection, $\kappa_M = Id - \pi_M$ fluctuation operator

$$(\nabla \cdot \boldsymbol{u}_h, q) \longrightarrow \tau_{u,gd,M}(\nabla \cdot \boldsymbol{u}_h, \nabla \cdot \boldsymbol{v}_h)_M$$
$$c_u(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) \longrightarrow \tau_{u,M}(\kappa_M((\boldsymbol{u}_M \cdot \nabla)\boldsymbol{u}_h), \kappa_M(\boldsymbol{u}_M \cdot \nabla \boldsymbol{v}_h))_M$$

# Analytical Results

$$\mathcal{U} := (\boldsymbol{u}, p), \qquad \mathcal{U}_h := (\boldsymbol{u}_h, p_h)$$

$$|||\mathcal{U}|||^2_{Stab} := \nu \|\nabla \boldsymbol{u}\|_0^2 + \sum_{M \in \mathcal{M}_h} \left( \gamma_M \|\nabla \cdot \boldsymbol{u}\|^2_{0,M} + \tau_M \|\kappa_M((\boldsymbol{u}_M \cdot \nabla)\boldsymbol{u})\|^2_{0,M} \right)$$

Aims:

- suitable choice of stabilization parameters
- quasi-optimal error estimates

$$\|\mathcal{U} - \mathcal{U}_h\|^2_{l^\infty(t_0,T;L^2(\Omega))} + |||\mathcal{U} - \mathcal{U}_h|||^2_{l^2(t_0,T;Stab)}$$
$$\leq C e^{C_G(T-t_0)} \left( \inf_{\mathcal{W}_h} \left\{ \|\mathcal{U} - \mathcal{W}_h\|^2_{l^\infty(t_0,T;L^2(\Omega))} + |||\mathcal{U} - \mathcal{W}_h|||^2_{l^2(t_0,T;Stab)} \right\} \right)$$

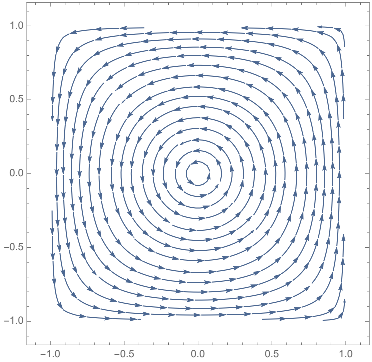
- semi-robust error estimates $C(\not{\nu})$, $C_G(\not{\nu})$

# Overview

# Analytical Test Case - Setting

$$\boldsymbol{u}(\boldsymbol{x},t) = \begin{pmatrix} -\cos\left(\frac{\pi}{2}x\right)\cdot\sin(\frac{\pi}{2}y)\cdot\sin(\pi\cdot t) \\ \sin\left(\frac{\pi}{2}x\right)\cdot\cos(\frac{\pi}{2}y)\cdot\sin(\pi\cdot t) \end{pmatrix}$$

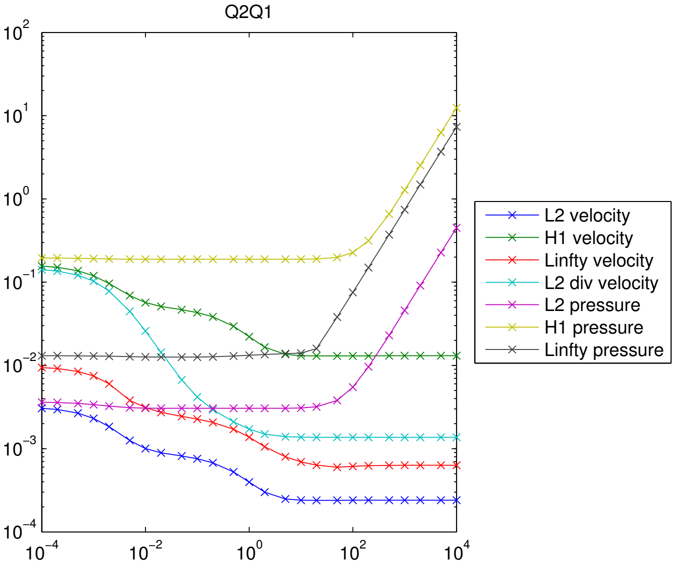
$$p(\boldsymbol{x},t) = -\pi\cdot\sin\left(\frac{\pi}{2}x\right)\cdot\sin\left(\frac{\pi}{2}y\right)\cdot\sin(\pi\cdot t)$$



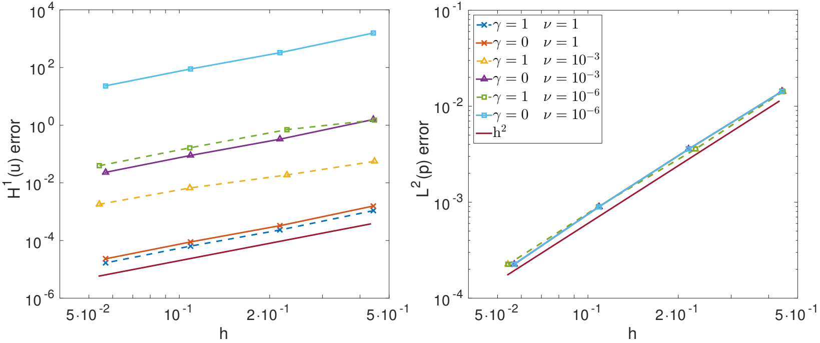

$$\frac{\|\boldsymbol{u}\|_{k+1}}{\|p\|_k} = 1 \qquad \Omega = [-1,1]^2$$

$$T = 4\cdot 10^{-1} \qquad \Delta t = 4\cdot 10^{-5}$$

# Analytical Test Case - Parameter Choice - $\nu = 10^{-4}$



# Analytical Test Case - Parameter Choice

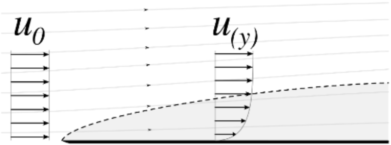


# Blasius Flow, Flow over Horizontal Plate

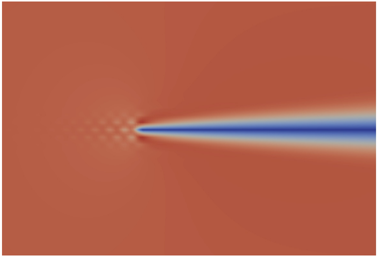
Prandtl's Boundary Layer Equation

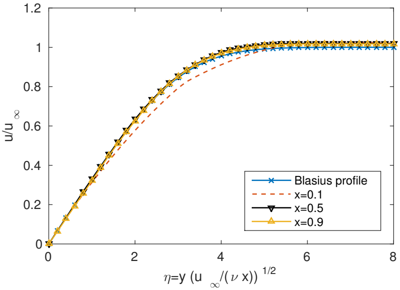
$$u = u_\infty f'(\eta)$$
$$2f'''(\eta) + f(\eta)f''(\eta) = 0$$
$$f(0) = f'(0) = 0,\ f'(\infty) = 1$$

where $\eta = y\sqrt{u_\infty/(\nu x)}$ is a non-dimensional variable.



# Blasius Flow

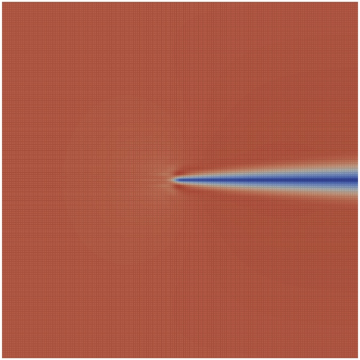





$$\nu = 10^{-3}, \quad \gamma = 1, \quad \tau_M = 0, \quad h = 2^{-5}$$

# Blasius Flow - $\nu = 10^{-3}$

Adaptively refined mesh, $\tau_M = 0$



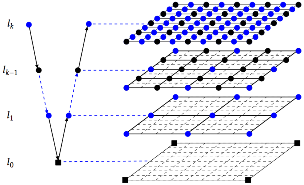
Globally refined mesh, $\tau_M = 1$

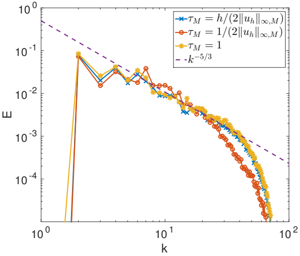
# Overview

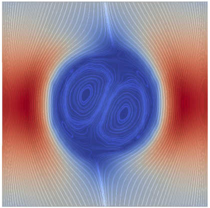
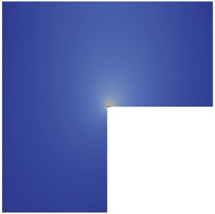
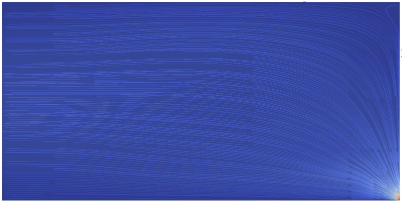
# Further Topics

Not covered today

- Time discretization
- Linear algebra
  - Fast solver
  - Multigrid
  - Matrix-Free methods
- Implementation ⇒ `deal.II`
- Turbulence modeling
- MHD, nonisothermal flow, . . .

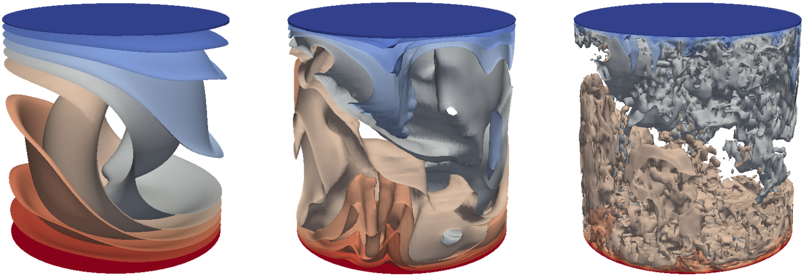


Thanks for your attention!

# Rayleigh-Bénard Convection



$Ra = 10^5$ $Ra = 10^7$ $Ra = 10^9$

$T = 1000$, $Pr = 0.786$, $N = 10 \cdot 16^3$, $\gamma_M = 0.1$