Schwarz Smoothers for Conforming Stabilized Discretizations of the Stokes Equations

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The Oseen problem

Oseen

Consider the Oseen problem

$$
\begin{aligned}
(\boldsymbol{f}, \boldsymbol{v}) = \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + \frac{\kappa}{2}\left((\boldsymbol{w} \cdot \nabla \boldsymbol{u}, \boldsymbol{v}) - (\boldsymbol{w} \cdot \nabla \boldsymbol{v}, \boldsymbol{u})\right) \\
- (p, \nabla \cdot \boldsymbol{v}) + (\nabla \cdot \boldsymbol{u}, q) + \gamma(\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v}) \\
\nabla \cdot \boldsymbol{w} = 0 \\
\nabla \cdot \boldsymbol{u} = 0
\end{aligned}
$$

with the grad-div stabilization term $\gamma(\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v})$.

This leads to the following structure of the system matrix:

$$
\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ 0 \end{pmatrix}
$$

Hence, we have to solve a symmetric (in case $\kappa = 0$), but indefinite problem.

Multigrid - V-Cycle

1. Pre-smoothing:

$$u^{(k+1)} = u^{(k)} - \mathcal{R}_l^{-1}(\mathcal{A}_l u^{(k)} - f_l), \qquad 0 \le k < m_{pre}$$

2. Coarse grid correction:

$$\begin{aligned} f_{l-1} &= \Pi_{l-1}^T (f_l - \mathcal{A}_l u^{(m_{pre})}) \\ v^{(k+1)} &= MG_{l-1}(v^{(k)}, f_{l-1}), \qquad 0 \le k < m_{coarse} \\ w^{(0)} &= u^{(m_{pre})} + v^{(m_{coarse})} \end{aligned}$$

3. Post-smoothing:

$$w^{(k+1)} = w^{(k)} - \mathcal{R}_l^{-1}(\mathcal{A}_l w^{(k)} - f_l), \qquad 0 \le k < m_{post}$$

4. Assign: $MG(u^{(0)}, f_l) = w^{(m_{post})}$

Coarse grid solver $MG_0(u(0), f) = \mathcal{A}_0^{-1} f_0$

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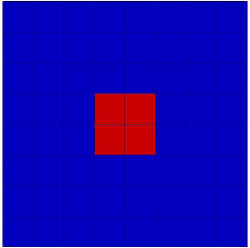
Additive Schwarz Smoother

Hermann Amandus Schwarz

- Take the local structure of the problem into account
- Use local problems for preconditioning

$$\mathcal{R}_l = \sum_{K \in \mathcal{T}_l} \mathcal{P}_K \mathcal{A}_K^{-1}$$

- Vertex patches



Raviart-Thomas Elements

Original result from Kanschat and Mao[1] using Raviart-Thomas elements.

Key assumption

$$\nabla \cdot V_h = Q_h \qquad V_{h,0}^{div} \subset \ldots \subset V_{h,L}^{div}$$

where

$$V_{h,l}^{div} := \{v_h \in V_l : (\nabla \cdot u_h, q_h) = 0 \quad \forall q_h \in Q_h\}$$

Can this assumption be weakened and the result be applied to other inf-sup stable elements?

[1] Guido Kanschat and Youli Mao. "Multigrid methods for Hdiv-conforming discontinuous Galerkin methods for the Stokes equations". In: *Journal of Numerical Mathematics* 23.1 (2015), pp. 51–66

Raviart-Thomas Elements

Theorem[2]

The multilevel iteration $I - \mathcal{B}_L \mathcal{A}_L$ for the Stokes problem

- with the variable V-cycle operator $\mathcal{B}_L$
- employing the smoother $\mathcal{R}_l$ with suitably small scaling factor η

is a contraction with contraction number independent of the level L.

[2] Guido Kanschat and Youli Mao. “Multigrid methods for Hdiv-conforming discontinuous Galerkin methods for the Stokes equations”. In: *Journal of Numerical Mathematics* 23.1 (2015), pp. 51–66

Helmholtz-like decomposition

Denote the bilinear form a_l corresponding to the weak Laplace operator by

$$a_l(\boldsymbol{u}, \boldsymbol{v}) := \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v})$$

For $\boldsymbol{u}_l \in \boldsymbol{V}_l$ define $\boldsymbol{u}_l^0 \in \boldsymbol{V}_h^{div} l$ as projection of $\boldsymbol{u}_l$ onto $\boldsymbol{V}_h^{div} l$ with respect to a_l, i.e.

$$a_l(\boldsymbol{u}_l^0, \boldsymbol{v}_l) = a_l(\boldsymbol{u}_l, \boldsymbol{v}_l) \quad \forall \boldsymbol{v}_l \in \boldsymbol{V}_h^{div} l.$$

Then define $\boldsymbol{u}_l^\perp$ by $\boldsymbol{u}_l^\perp := \boldsymbol{u}_l - \boldsymbol{u}_l^0$.

Lemma

$$\frac{\alpha}{d^2}\|\nabla \cdot \boldsymbol{u}_l^\perp\|_0^2 \leq a_l(\boldsymbol{u}_l^\perp, \boldsymbol{u}_l^\perp) \leq \frac{\nu}{\gamma_l^2}\|\pi_{Q_h}(\nabla \cdot \boldsymbol{u}_l^\perp)\|_0^2$$

Stokes, Perturbed Primal and Perturbed Dual Problem

Idea: Eliminate the pressure by considering a perturbed formulation

$$\alpha(\boldsymbol{u}_l, \boldsymbol{v}_h) + \nu(\nabla \boldsymbol{u}_l, \nabla \boldsymbol{v}_h) + \gamma(\nabla \cdot \boldsymbol{u}_h - \epsilon p_h, \nabla \cdot \boldsymbol{v}_h - \epsilon q_h)$$
$$-(p_l, \nabla \cdot \boldsymbol{v}_h) + (\nabla \cdot \boldsymbol{u}_l, q_h) - \epsilon(p_l, q_h) = (\boldsymbol{f}, \boldsymbol{v}_h)$$

Defining the operator $\mathcal{A}_l : \boldsymbol{V}_l \times Q_l \to (\boldsymbol{V}_l \times Q_l)^*$ by

$$\begin{aligned}\mathcal{A}_l((\boldsymbol{u}_l, p_l), (\boldsymbol{v}_h, q_h)) :=& \alpha(\boldsymbol{u}_l, \boldsymbol{v}_h) + \nu(\nabla \boldsymbol{u}_l, \nabla \boldsymbol{v}_h) \\ &+ \gamma(\nabla \cdot \boldsymbol{u}_l - \epsilon p_l, \nabla \cdot \boldsymbol{v}_h - \epsilon q_h) \\ &+ (p_l, \nabla \cdot \boldsymbol{v}_h) + (\nabla \cdot \boldsymbol{u}_l, q_h) - \epsilon(p_l, q_h).\end{aligned}$$

this problem can be written as $\mathcal{A}_l((\boldsymbol{u}_l, p_l), (\boldsymbol{v}_h, q_h)) = (\boldsymbol{f}, \boldsymbol{v}_h)$ for all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_l \times Q_l$.

Stokes, Perturbed Primal and Perturbed Dual Problem

For $\epsilon > 0$, the Stokes problem can be rewritten as

$$\begin{aligned}\mathcal{A}_l(\boldsymbol{u}_l, \boldsymbol{v}_h) &:= \alpha(\boldsymbol{u}_l, \boldsymbol{v}_h) + \nu(\nabla \boldsymbol{u}_l, \nabla \boldsymbol{v}_h) \\ &\quad + \gamma(\pi_{Q_h}^{\perp}(\nabla \cdot \boldsymbol{u}_l), \pi_{Q_h}^{\perp}(\nabla \cdot \boldsymbol{v}_h)) \\ &\quad + \frac{1}{\epsilon}(\pi_{Q_h}(\nabla \cdot \boldsymbol{u}_l), \pi_{Q_h}(\nabla \cdot \boldsymbol{v}_h)). \\ \mathcal{A}_l(\boldsymbol{u}_l, \boldsymbol{v}_h) &= (\boldsymbol{f}, \boldsymbol{v}_h)\end{aligned}$$

for all $\boldsymbol{v}_h \in \boldsymbol{V}_l$.

Lemma

Let $(\boldsymbol{u}_l, p_l)$ be the solution to the perturbed problem in two variables and $\boldsymbol{u}_l$ the solution to the perturbed problem in one variable. Then it holds

$$\boldsymbol{u}_l = \boldsymbol{u}_l \qquad \epsilon p_l = \pi_{Q_h}(\nabla \cdot \boldsymbol{u}_l) = \pi_{Q_h}(\nabla \cdot \boldsymbol{u}_l)$$

Convergence of the Perturbation

Let $(\boldsymbol{u}, p)$ be the solution to the continuous Stokes problem and $(\boldsymbol{u}_h, p_h)$ the solution to the discretized (perturbed) problem.

Lemma

It holds

$$\alpha\|\boldsymbol{u}-\boldsymbol{u}_h\|_0^2+\nu\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_0^2+\gamma\|\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h)\|_0^2+\|p-p_h\|_0^2$$
$$\lesssim \epsilon+h^{2k_p+2}+h^{2k_u}.$$

Convergence Result

Assumptions

If $\mathcal{R}_l$ satisfies for all $\boldsymbol{w} \in \boldsymbol{V}_l$

$$\mathcal{A}_l((\mathcal{I}_l - \mathcal{R}_l\mathcal{A}_l)\boldsymbol{w}, \boldsymbol{w}) \geq 0 \tag{1}$$

$$(\mathcal{R}_l^{-1}[\mathcal{I}_l - \mathcal{P}_{l-1}]\boldsymbol{w}, [\mathcal{I}_l - \mathcal{P}_{l-1})\boldsymbol{w}) \leq \beta_l\mathcal{A}_l([\mathcal{I}_l - \mathcal{P}_{l-1}]\boldsymbol{w}, [\mathcal{I}_l - \mathcal{P}_{l-1}]\boldsymbol{w}) \tag{2}$$

where $\beta_l = \mathcal{O}(\gamma_l^{-1})$, then it holds

$$0 \leq \mathcal{A}_l([\mathcal{I}_l - \mathcal{B}_l\mathcal{A}_l)\boldsymbol{w}, \boldsymbol{w}) \leq \delta\mathcal{A}_l(\boldsymbol{w}, \boldsymbol{w}), \quad \forall \boldsymbol{w} \in \boldsymbol{V}_l$$

where $\delta < 1$.

Lemma

Let $\eta \leq 2^{-dim}$*, then*

$$\mathcal{A}_l((\mathcal{I}_l - \mathcal{R}_l\mathcal{A}_l)\boldsymbol{w}, \boldsymbol{w}) \geq 0, \quad \forall \boldsymbol{w} \in \boldsymbol{V}_l.$$

Stable decomposition

Lemma

For all $\boldsymbol{w} \in \boldsymbol{V}_l$ it holds

$$(\mathcal{R}_l^{-1}[\mathcal{I}_l - \mathcal{P}_{l-1}]\boldsymbol{w}, [I - \mathcal{P}_{l-1})\boldsymbol{w}) \le \beta_l \mathcal{A}_l([\mathcal{I}_l - \mathcal{P}_{l-1}]\boldsymbol{w}, [\mathcal{I}_l - \mathcal{P}_{l-1}]\boldsymbol{w})$$

Essentially, we only need to find a decomposition $(\boldsymbol{u}_v)_v$ of $[\mathcal{I}_l - \mathcal{P}_{l-1}]\boldsymbol{w}$, i.e.

$$\boldsymbol{u} := [\mathcal{I}_l - \mathcal{P}_{l-1}]\boldsymbol{w} = \sum_v \mathcal{I}_{l,v}\boldsymbol{u}_v.$$

such that

$$\sum_v (\mathcal{A}_l \mathcal{I}_{l,v}\boldsymbol{u}_v, \mathcal{I}_{l,v}\boldsymbol{u}_v) \le \beta_l \mathcal{A}_l([\mathcal{I}_l - \mathcal{P}_{l-1}]\boldsymbol{w}, [\mathcal{I}_l - \mathcal{P}_{l-1}]\boldsymbol{w}).$$

Stable decomposition

Theorem

For any $\mathbf{v}_l \in \mathbf{V}_l$ there exists a decomposition $\mathbf{v}_{l,j}$ such that

$$\sum_{j=0}^{J} \mathcal{A}_l(\mathbf{v}_{l,j}, \mathbf{v}_{l,j}) \lesssim \mathcal{A}_l(\mathbf{v}_l, \mathbf{v}_l)$$

provided $\tau_{gd} \lesssim \min\{\nu, \epsilon^{-1}\}$.

Assumption

$$\sum_{v} a_l(\mathbf{u}_v^\perp, \mathbf{u}_v^\perp) \leq C a_l(\mathbf{u}_l^\perp, \mathbf{u}_l^\perp)$$

This clearly holds, for discontinuous, divergence-free elements. What about TH?

Decomposition of the divergence

For discontinuous pressure spaces we first notice

$$\sum_v \boldsymbol{V}_{l,v} = \boldsymbol{V}_l$$

and for every decomposition it holds

$$\begin{aligned}
\boldsymbol{v}_l \in \boldsymbol{V}_l^{div} \quad &\Longleftrightarrow & (\boldsymbol{v}_l, q_l) = 0 \qquad & \forall q_l \in Q_l \\
&\Longleftrightarrow & \Big(\sum_v v_{l,v}, q_{l,K}\Big) = 0 \qquad & \forall K \in \Omega_l, \quad q_{l,K} \in Q_{l,K} \\
&\Longleftarrow & (v_{l,v}, q_{l,K}) = 0 \qquad & \forall v, \quad \forall K \in \Omega_{l,v}, \quad q_{l,K} \in Q_{l,K} \\
&\Longleftrightarrow & v_{l,v} \in \boldsymbol{V}_{l,v}^{div} \qquad & \forall v
\end{aligned}$$

which means $\sum_v \boldsymbol{V}_{l,v}^{div} \subset \boldsymbol{V}_l^{div}$.

Feng & Lorton

Following Feng & Lorton[3] we need to consider the assumptions

Assumption

- There exists a positive constant C_a such that

$$|a(v,w)| \leq C_a \|v\|_V \|w\|_W \quad \forall v \in V, w \in W.$$

- There exists positive constants γ_a, β_a such that

$$\sup_{w \in W} \frac{a(v,w)}{\|w\|_W} \leq \gamma_a \|v\|_V \quad \forall v \in V,$$
$$\sup_{v \in V} \frac{a(v,w)}{\|v\|_V} \leq \beta_a \|w\|_W \quad \forall w \in W.$$

Which follow by standard techniques for the considered case.

[3] Xiaobing Feng and Cody Lorton. "On Schwarz Methods for Nonsymmetric and Indefinite Problems". In: *arXiv preprint arXiv:1308.3211* (2013)

Norms

We need to consider the norm

$$\|(\boldsymbol{u}, p)\|_a = \sup_{(\boldsymbol{v}, q)} \frac{a((\boldsymbol{u}, p), (\boldsymbol{v}, q))}{\|(\boldsymbol{v}, q)\|_{V \times Q}}$$

and $\|(\boldsymbol{v}, q)\|$ defined via

$$\|(\boldsymbol{v}, q)\|^2 := \nu \|\nabla \boldsymbol{v}\|_0^2 + \gamma \|\nabla \cdot \boldsymbol{u}\|_0^2 + \|q\|_0^2.$$

Norms

In particular, we have

$$
\begin{aligned}
&a((\boldsymbol{u},p),(\boldsymbol{v},q))\\
&\lesssim \|(\boldsymbol{u},p)\|\|(\boldsymbol{v},q)\|\left(1+\kappa\|\boldsymbol{w}\|_{L^\infty}+\min\left\{\frac{1}{\sqrt{\gamma}},\frac{1}{\sqrt{\nu}}\right\}\right)\\
&\lesssim \|(\boldsymbol{u},p)\|\|(\boldsymbol{v},q)\|_{V\times Q}\left(1+\sqrt{\frac{\gamma}{\nu}}\right)\left(1+\kappa\|\boldsymbol{w}\|_{L^\infty}+\min\left\{\frac{1}{\sqrt{\gamma}},\frac{1}{\sqrt{\nu}}\right\}\right)
\end{aligned}
$$

$$
\|(\boldsymbol{u},p)\|^2 = a((\boldsymbol{u},p),(\boldsymbol{v},q)) \le \sup_{(\boldsymbol{v},q)} \frac{a((\boldsymbol{u},p),(\boldsymbol{v},q))}{\|(\boldsymbol{v},q)\|}
$$

due to the inf-sup stability of the chosen discrete spaces.

Hence, these norms are equivalent for $\gamma \lesssim \nu$.

Stable Decomposition

Now, Feng & Lorton require a energy stable decomposition

$$\sum_{v} \|(\boldsymbol{u}_v, p_v)\|_{a_j} \leq C\|(\boldsymbol{u}_v, p_v)\|_a$$

Equivalence of the norms $\Rightarrow$ Proofing for the energy norm sufficient

- standard techniques as before
- requires $\gamma \lesssim \nu$ in general

$\Longrightarrow$ The condition number $\kappa_a(P_{ad})$

$$\kappa_a(P_{ad}) := \|P_{ad}\|_a \|P_{ad}^{-1}\|_a$$

of the two-level preconditioner defined by the local Schwarz smoothers is bounded.

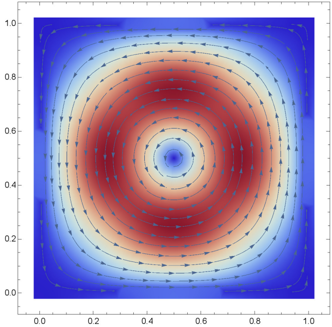
Numerical Results - Test Problem

We consider the test problem

$$-\nu\Delta u + \nabla p = -\nu\Delta u_{ref} + \nabla p_{ref}$$
$$\nabla \cdot u = 0$$

with the reference solution

$$\boldsymbol{u}(x,y) = \begin{pmatrix} \sin(\pi x)\sin(\pi x)\sin(2\pi y)\pi/2 \\ -\sin(\pi y)\sin(\pi y)\sin(2\pi x)\pi/2 \end{pmatrix}$$
$$p(x,y) = \sin(\pi x)\cos(\pi y).$$



Observe for $\nu = 10^{-6}$

- errors
- iteration counts (error reduction by 10^{-6}).

Numerical Results - Optimal Stabilization Parameter

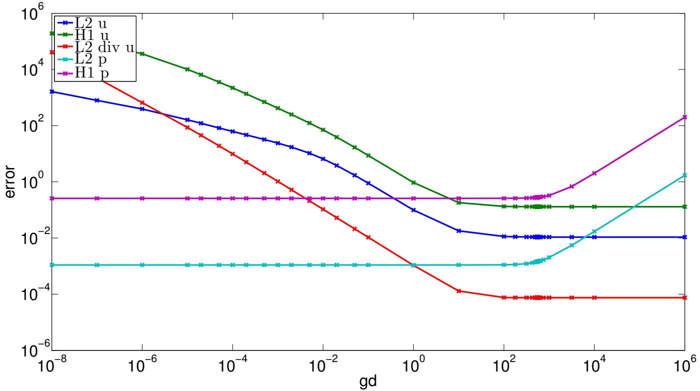


Figure : $\mathbb{Q}_2/\mathbb{Q}_1$ elements, optimal $\gamma = 544.917$, $\nu = 10^{-6}$

Numerical Results - Optimal Stabilization Parameter

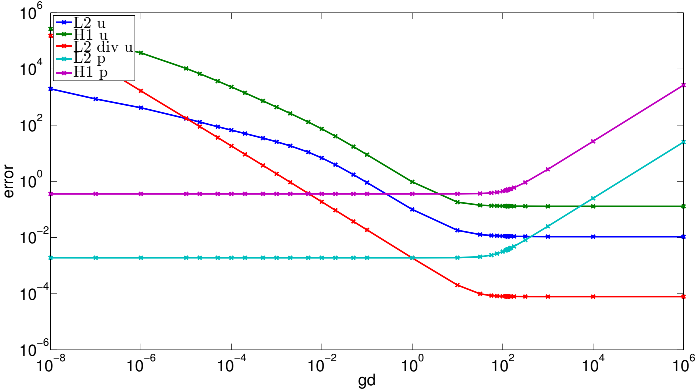


Figure : $\mathbb{Q}_2/\mathbb{P}_1^-$ elements, optimal $\gamma = 128.93$, $\nu = 10^{-6}$

Numerical Results - Optimal Stabilization Parameter

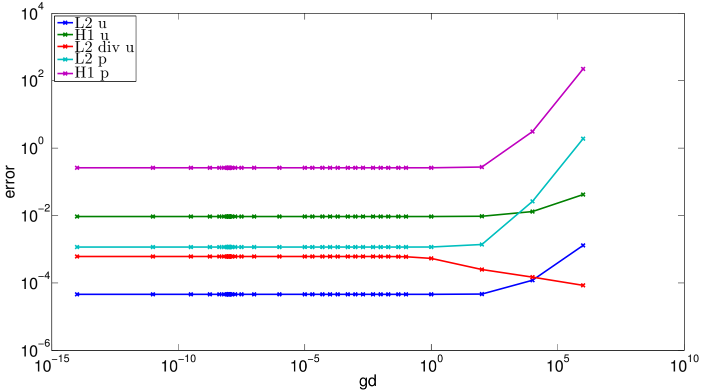


Figure : $\mathbb{Q}_2/\mathbb{Q}_1$ elements, optimal $\gamma = 1.e - 8, \nu = 1$

Numerical Results - Optimal Stabilization Parameter

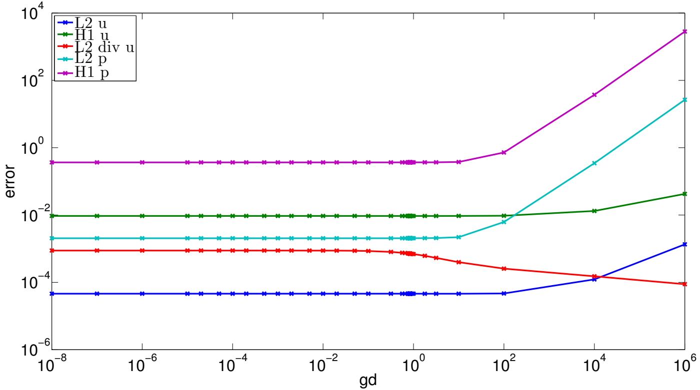


Figure : $\mathbb{Q}_2/\mathbb{P}_1^-$ elements, optimal $\gamma = 0.841, \nu = 1$

Numerical Results

Element	ν	Refinement 1	2	3	4	5	6
$\mathbb{Q}_2/\mathbb{P}_1^-$	10^0	$1 \cdot 10^{-7}$	$1 \cdot 10^{-8}$	$5 \cdot 10^{-4}$	$3 \cdot 10^{-3}$	$8 \cdot 10^{-1}$	$3 \cdot 10^{0}$
$\mathbb{Q}_2/\mathbb{Q}_1$	10^0	$5 \cdot 10^{-1}$	$5 \cdot 10^{-1}$	$1 \cdot 10^{-1}$	$1 \cdot 10^{-8}$	$1 \cdot 10^{-8}$	$1 \cdot 10^{-8}$
$\mathbb{Q}_2/\mathbb{P}_1^-$	10^{-6}	$1 \cdot 10^{0}$	$3 \cdot 10^{1}$	$2 \cdot 10^{1}$	$5 \cdot 10^{1}$	$1 \cdot 10^{2}$	$9 \cdot 10^{1}$
$\mathbb{Q}_2/\mathbb{Q}_1$	10^{-6}	$9 \cdot 10^{0}$	$9 \cdot 10^{1}$	$9 \cdot 10^{1}$	$2 \cdot 10^{2}$	$5 \cdot 10^{2}$	$4 \cdot 10^{2}$
$\mathbb{Q}_3/\mathbb{P}_2^-$	10^0	$1 \cdot 10^{-6}$	$1 \cdot 10^{-5}$	$2 \cdot 10^{-3}$	$5 \cdot 10^{-1}$	$2 \cdot 10^{0}$	$7 \cdot 10^{1}$
$\mathbb{Q}_3/\mathbb{Q}_2$	10^0	$5 \cdot 10^{-4}$	$2 \cdot 10^{-2}$	$1 \cdot 10^{0}$	$2 \cdot 10^{0}$	$2 \cdot 10^{0}$	$1 \cdot 10^{0}$
$\mathbb{Q}_3/\mathbb{P}_2^-$	10^{-6}	$1 \cdot 10^{5}$	$5 \cdot 10^{3}$	$4 \cdot 10^{0}$	$1 \cdot 10^{1}$	$5 \cdot 10^{2}$	$9 \cdot 10^{-1}$
$\mathbb{Q}_3/\mathbb{Q}_2$	10^{-6}	$1 \cdot 10^{5}$	$8 \cdot 10^{2}$	$6 \cdot 10^{1}$	$4 \cdot 10^{1}$	$4 \cdot 10^{2}$	$5 \cdot 10^{-1}$

Table : Optimal stabilization parameter

#Levels	0	1	2	3	4
RT1, 2D	3	9	10	11	13
RT2, 2D	3	9	10	11	11
RT1, 3D	3	13	16	20	

Table : Iteration counts for Raviart-Thomas elements

Numerical Results - Iteration Counts - $\mathbb{Q}_2/\mathbb{P}_1^-$ - $\eta = \frac{1}{4}$ - 2D

#Levels	τ_{gd} 10^{0}	10^{-1}	10^{-2}	10^{-3}	10^{-4}
0	2	2	2	2	2
1	24	23	27	22	17
2	70	65	57	43	25
3	236	159	93	48	27
4	459	247	105	49	28

#Levels	τ_{gd} 10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}
0	2	2	2	2	2
1	14	12	13	13	13
2	19	18	18	18	18
3	19	19	19	19	19
4	19	18	19	20	20

Numerical Results - Iteration Counts - $\mathbb{Q}_2/\mathbb{Q}_1$ - $\eta = \frac{1}{8}$ - 2D

#Levels	τ_{gd} 10^{0}	10^{-1}	10^{-2}	10^{-3}	10^{-4}
0	2	2	2	2	2
1	35	31	36	28	22
2	137	98	85	59	31
3	454	294	159	71	37
4	-	610	190	76	38

#Levels	τ_{gd} 10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}
0	2	2	2	2	2
1	17	17	17	17	17
2	23	24	24	25	25
3	28	31	34	35	35
4	28	33	38	39	39

Numerical Results - Iteration Counts - $\mathbb{Q}_3/\mathbb{P}_2^-$ - $\eta = \frac{1}{4}$ - 2D

#Levels	τ_{gd} 10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}
0	2	2	2	2	2
1	12	12	13	14	15
2	19	19	19	20	17
3	30	30	29	24	16
4	39	38	35	24	16

#Levels	τ_{gd} 10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}
0	2	2	3	3	2
1	15	16	16	16	16
2	15	17	18	18	18
3	15	17	18	18	18
4	14	16	18	18	18

Numerical Results - Iteration Counts - $\mathbb{Q}_3/\mathbb{Q}_2$ - $\eta = \frac{1}{8}$ - 2D

	τ_{gd}				
#Levels	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}
0	2	2	2	2	2
1	19	19	20	20	21
2	31	31	31	27	25
3	36	35	33	32	25
4	50	50	50	36	25

	τ_{gd}				
#Levels	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}
0	2	2	2	2	2
1	21	22	22	22	22
2	25	28	31	31	31
3	26	30	33	34	34
4	27	32	37	37	38

Numerical Results - Iteration Counts - $\mathbb{Q}_2/\mathbb{P}_1^-$ - $\eta = \frac{1}{8}$ - 3D

	τ_{gd}				
#Levels	10^{0}	10^{-1}	10^{-2}	10^{-3}	10^{-4}
0	2	2	2	2	2
1	72	64	59	40	24
2	426	264	146	65	37
3	928	402	149	67	37

	τ_{gd}				
#Levels	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}
0	2	2	2	2	2
1	21	20	21	21	21
2	26	28	29	30	30
3	27	29	31	31	31

Summary

Results

- Local schwarz smoothers applicable for inf-sup stable conforming elements
- Comparable results to Raviart-Thomas elements
- $\mathbb{Q}_{k+1}/\mathbb{P}_k^-$ elements perform much better than $\mathbb{Q}_{k+1}/\mathbb{Q}_k$ elements
- Analysis requires $\tau_{gd} \lesssim \nu$; sharpness confirmed by numerical results
- Positive effect of stabilization especially for $\mathbb{Q}_{k+1}/\mathbb{Q}_k$ elements

Outlook/Challenges:

- Consider also convection dominated problems (Oseen, Navier-Stokes)
- Lift the restriction $\tau_{gd} \lesssim \nu$
- Complete the multigrid analysis

Thank you for your attention!

Numerical Results - Iteration Counts

- 2D
- $\nu = 10^{-6}$
- multiplicative smoother
- with smoother relaxation term of 1.

	$\mathbb{Q}_2 \times \mathbb{Q}_1$			$\mathbb{Q}_2 \times \mathbb{P}_1^-$			$Q_2^+ \times \mathbb{Q}_1$			$\mathbb{Q}_2 \times (\mathbb{Q}_1 + \mathbb{Q}_0)$		
	γ			γ			γ			γ		
GR	0.0	10^{-6}	1.0	0.0	10^{-6}	1.0	0.0	10^{-6}	1.0	0.0	10^{-6}	1.0
0	1	1	1	1	1	1	1	1	1	1	1	1
1	6	6	16	3	3	9	18	17	38	7	6	16
2	9	8	49	5	5	32	28	34	97	21	19	58
3	10	9	138	6	5	89	37	40	553	65	60	381
4	11	9	282	6	5	195	38	41	1000f	-	-	-

Numerical Results - Iteration Counts

- 2D
- $\nu = 10^{-6}$
- multiplicative smoother
- with smoother relaxation term of 1.0 for all elements

	$\mathbb{Q}_3 \times \mathbb{Q}_2$			$\mathbb{Q}_3 \times \mathbb{P}_2^-$			$Q_3^+ \times \mathbb{Q}_2$			$\mathbb{Q}_3 \times (\mathbb{Q}_2 + \mathbb{Q}_0)$		
	γ			γ			γ			γ		
GR	0.0	10^{-6}	1.0	0.0	10^{-6}	1.0	0.0	10^{-6}	1.0	0.0	10^{-6}	1.0
0	1	1	1	1	1	1	1	1	1	1	1	1
1	5	5	5	3	3	3	16	16	27	7	8	6
2	9	9	10	4	4	6	32	35	44	17	16	12
3	12	11	18	4	3	8	39	41	76	31	28	22
4	13	11	31	3	3	8	46	44	156	57	50	37

Numerical Results - Iteration Counts

- 3D
- $\nu = 1^{-6}$
- additive smoother
- with smoother relaxation term of .25 for all elements

	$\mathbb{Q}_2 \times \mathbb{Q}_1$			$\mathbb{Q}_2 \times \mathbb{P}_1^-$			$Q_2^+ \times \mathbb{Q}_1$			$\mathbb{Q}_2 \times (\mathbb{Q}_1 + \mathbb{Q}_0)$		
	γ			γ			γ			γ		
GR	0.0	10^{-6}	1.0	0.0	10^{-6}	1.0	0.0	10^{-6}	1.0	0.0	10^{-6}	1.0
0	2	2	2	2	2	2	2	2	5	2	2	2
1	35	34	477	21	20	72	183	177	1000f	38	32	194
2	1000f	1000f	1000f	30	38	426	1000f	1000f	1000f	1000f	1000f	1000f