IWR – Universität Heidelberg
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Due date: **17.1.2014**

Homework No. 10
Numerical Methods for PDE, Winter 2013/14

Problem 10.1: Condition number of the stiffness matrix I (15 points)

Consider the homogeneous Dirichlet problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

The problem is solved by a linear finite element method on a quasi-uniform triangular mesh.

(a) Show that the stiffness matrix is symmetric and positive definite.

(b) Show that the maximal eigenvalue of the stiffness matrix A_h is bounded by a term of order $\mathcal{O}(h^{d-2})$. Use an example to show that this bound is asymptotically optimal.

(c) Show that the minimal eigenvalue of the stiffness matrix A_h is bounded from below by a term of order $\mathcal{O}(h^d)$.

Hint: the ellipticity of the bilinear form may help.

(d) Use an example to show that this bound is asymptotically optimal.

Hint: you may use the following fact: the eigenfunction to the lowest eigenvalue of the operator A associated withthe bilinear form $a(.,.)$ is smooth and therefore, it can be well approximated by finite element functions.

(e) Conclude that we have for the spectral condition number of the matrix

$$\kappa(A_h) \simeq h^{-2}.$$

(f) What changes if we consider quadratic finite elements instead of linear ones?