IWR – Universität Heidelberg
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Due date: **20.12.2013**

Homework No. 8
Numerical Methods for PDE, Winter 2013/14

Problem 8.1: Dual Problem and Error Functionals (8 points)

Consider the following convection-diffusion equation

$$\begin{aligned} -\varepsilon\Delta u + \beta\cdot\nabla u &= f &&\text{in } \Omega,\\ u &= 0 &&\text{on } \partial\Omega, \end{aligned}$$

with a sufficiently smooth β and $\varepsilon > 0$. The domain Ω is a convex polygon in two space dimensions.

(a) Formulate the dual problem weakly as well as classically for a general linear error functional $J(\cdot)$.

(b) Draw the transport fields of the dual solution for the specific transport fields

(i) $\beta_1 = (1,1)^T$,
(ii) $\beta_2 = (y-0.5, 0.5-x)^T$.

(c) Formulate appropriate error functionals $J(\cdot)$ for the computation of the

(i) energy error $\|\nabla(u-u_h)\|_{L^2(\Omega)}$,
(ii) L^2-error $\|u-u_h\|_{L^2(\Omega)}$,
(iii) the mean value of the solution u.

Problem 8.2: "Sharp" L^2-A Posteriori Error Estimate (8 points)

Consider the standard Poisson problem

$$\begin{aligned} -\Delta u(x) &= f &&\text{in } \Omega,\\ u(x) &= 0 &&\text{on } \partial\Omega, \end{aligned}$$

where Ω is a convex polygonal domain in $\mathbb{R}^2$.

The a posteriori error estimate in L^2 is given by

$$\|e_h\|_{L^2(\Omega)} \le \eta_{L^2}(u_h) := c\left(\sum_{T\in\Omega_h} h_T^4\left(\rho_T(u_h)^2 + h_T^{-1}\rho_{\partial T}(u_h)^2\right)\right)^{\frac{1}{2}},$$

with $e_h = u - u_h$, and

$$\rho_T(u_h) := \|f + \Delta u_h\|_{L^2(T)}, \qquad \rho_{\partial T}(u_h) = \frac{1}{2}\|[\partial_n u_h]\|_{L^2(\partial T)},$$

where $[\cdot]$ denotes the jump of the normal derivative on the edge of neighboring cells.

(a) Prove the estimate

$$\|\partial_n u\|^2_{L^2(\partial T)} \le c\left(h_T\|\Delta u\|^2_{L^2(T)} + h_T^{-3}\|u\|^2_{L^2(T)}\right),$$

by proving a similar estimate on the reference element.

Hint: Use the regularity theory for elliptic PDE's and the trace equation.

(b) Use the estimate to prove that $\eta_{L^2}(u_h)$ is reliable and efficient in the following sense

$$\eta_{L^2}(u_h) \le c_1\|e_h\|_{L^2(\Omega)} + c_2 h^2\|f\|_{L^2(\Omega)}$$

with $h := \max_{T\in\Omega_h} h_T$.