IWR – Universität Heidelberg
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Due date: **29.11.2013**

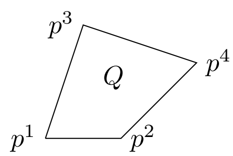
Homework No. 5
Numerical Methods for PDE, Winter 2013/14

Problem 5.1: Transformation of Quadrilaterals (6 points)

Transformation from the reference square $\widehat{Q} = [0,1]^2$ to a general quadrilateral given by vertices $p^i = (x_i, y_i)^T$ for $i = 1, \dots, 4$ can be obtained by the mapping F with

$$F(\boldsymbol{\xi}) = p^1(1-\xi)(1-\eta) + p^2(1-\eta)\xi + p^3\eta(1-\xi) + p^4\xi\eta.$$

Here, $\boldsymbol{\xi} = (\xi, \eta)^T$. The order of vertices follows the scheme



(a) Show that indeed $Q = F(\widehat{Q})$.

(b) Compute $\nabla F(\boldsymbol{\xi})$.

(c) Compute the Jacobi determinant $J(\xi)$ and show that $J(\xi) \geq 0$, if and only if the quadrilateral is convex.

(d) Compute the eigenvalues of $(\nabla F(\boldsymbol{\xi}))^T \nabla F(\boldsymbol{\xi})$ and relate them to $\|\nabla F(\boldsymbol{\xi})\|$ and $\|\nabla(F(\boldsymbol{\xi}))^{-1}\|$.

(e) What happens to $J(\xi)$, $\|\nabla F(\boldsymbol{\xi})\|$ and $\|(\nabla F(\boldsymbol{\xi}))^{-1}\|$ at p^1 if p^2 gets close to p^1?

Problem 5.2: Trace inequality for polynomials (4 points)

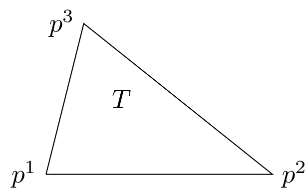
Let K be a shape regular triangle of diameter h. Show that for any polynomial $p \in P_k$ holds

$$\|p\|_{\partial K} \leq Ch^{-\frac{1}{2}} \|p\|_K$$
$$\|p\|_{\partial K} \leq Ch^{\frac{1}{2}} \|\nabla p\|_K$$

with constants C depending on the degree k and the constant of shape regularity.

Problem 5.3: Connection between the Shape of Triangles and the Stiffness Matrix (4 points)

We discretize the Poisson problem by piecewise linear finite elements on a triangulation of the unite square. Consider an arbitrary triangle T in this triangulation.



The nodal basis functions φ_i have the properties $\varphi_i(p_j) = \delta_{ij}$, $i, j = 1, 2, 3$.

(a) Show that

$$(\nabla\varphi_i, \nabla\varphi_i)_T > 0 \qquad \text{and} \qquad (\nabla\varphi_i, \nabla\varphi_j)_T \leq 0, \qquad i \neq j$$

as long as all interior angles are equal or smaller than $\frac{\pi}{2}$.

(b) The entries of the stiffness matrix are given by

$$a_{ij} = \sum_{T \in \mathcal{T}_h} (\nabla\varphi_i, \nabla\varphi_j)_T \,.$$

Conclude from the first part of the exercise that the diagonal entries of the matrix are always positive and the off-diagonal entries are smaller or equal to zero.

(c) **Bonus (3 points):** Show that furthermore the conditions

$$\sum_{j \neq i} |a_{ij}| \leq a_{ii}, \qquad \sum_{j \neq i^\star} |a_{i^\star j}| < a_{i^\star i^\star} \qquad \text{for a fixed } i^\star$$

are fulfilled. Discuss the properties of the matrix.