IWR – Universität Heidelberg
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Due date: **15.11.2013**

Homework No. 3
Numerical Methods for PDE, Winter 2013/14

Problem 3.1: Trilinearform

For $\Omega \subset \mathbb{R}^2$ consider the term

$$c(w;u,v) = (w \cdot \nabla u, v)\,, \quad c(\cdot;\cdot,\cdot) : W \times V \times V \to \mathbb{R}, \qquad \text{with } W = H^1_0(\Omega;\mathbb{R}^2), \quad \text{and} \quad V = H^1_0(\Omega).$$

Note: The semicolon indicates that we will later on use w as data, such that $c(w;.,.)$ is a bilinearform which fits into our existing framework.

(a) Show that $c(\cdot;\cdot,\cdot)$ is linear in each variable (altogether trilinear) and continuous.
Hint: Show that $|c(w;u,v)|$ is bounded. Use the Sobolev embedding theorem.

(b) Show that the following identity holds:

$$(w \cdot \nabla u, u) = -\frac{1}{2}\left(\nabla\cdot w, u^2\right).$$

Hint: The notation

$$\nabla\cdot\varphi := \partial_{x_1}\varphi_1 + \partial_{x_2}\varphi_2$$

denotes the divergence of a sufficiently regular vector field $\varphi(x) = \big(\varphi_1(x), \varphi_2(x)\big)^T$.

(c) Deduce an analogous formula for $w \in H^1(\Omega;\mathbb{R}^2)$ and $u \in H^1(\Omega)$ without zero boundary conditions.

Problem 3.2: Galerkin equations

Starting point is the one-dimensional problem

$$-u'' + u = f \quad \text{in } \Omega = (0,1),$$

for the space $V = H^1_0(\Omega)$.

We consider the equidistant mesh

$$x_j = jh, \quad j = 0,\dots,N, \quad \text{with } h = \frac{1}{N}$$

on the interval Ω with N mesh cells $I_j = (x_{j-1}, x_j)$. The finite-dimensional subspace V_h is now the span of piecewise linear functions

$$\varphi_j(x) = \begin{cases} \frac{x - x_{j-1}}{h}, & \text{if } x \in (x_{j-1}, x_j], \\ \frac{x_{j+1} - x}{h}, & \text{if } x \in (x_j, x_{j+1}), \\ 0, & \text{otherwise.} \end{cases}$$

where $j = 1,\dots,N-1$.

(a) Sketch the domain Ω with its subdivision and a reasonable number of functions φ_j.

(b) Argue that the space V_h contains exactly all functions which are linear on each cell I_j, continuous on Ω and have zero boundary conditions.

(c) Set up the Galerkin equations.

(d) Compute the 2×2 cell matrices A_k and cell vectors b_K.

(e) Assemble the cell matrices and cell vectors into the global matrix A and the global vector b.

(f) **Bonus (2 points):** Calculate the solution of Galerkin equations for the right hand side $f(x) = 1$ and plot it for $N = 5, 10, 20, 40, 100$ in one figure.

Each problem 6 points.